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Artistic Patterns in Hyperbolic Geometry

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Abstract

From antiquity, humans have created 2-dimensional art on flat surfaces (the Euclidean plane) and on surfaces of spheres. However, it wasn't until recently that they have created art in the third "classical geometry", the hyperbolic plane. M. C. Escher was the first person to do so, doing all the needed constructions laboriously by hand. To exhibit the true hyperbolic nature of such art, the pattern must exhibit symmetry and repetition. Thus, it is natural to use a computer to avoid the tedious hand constructions performed by Escher. We show a number of hyperbolic patterns, which are created by combining mathematics, artistic considerations, and computer technology.

Introduction

More than 100 years ago mathematicians created the first repeating patterns of the hyperbolic plane, triangle tessellations (see Figure 1) which were attractive, although not originally created for artistic purposes. In the late 1950's, the Dutch artist M. C. Escher became the first person to

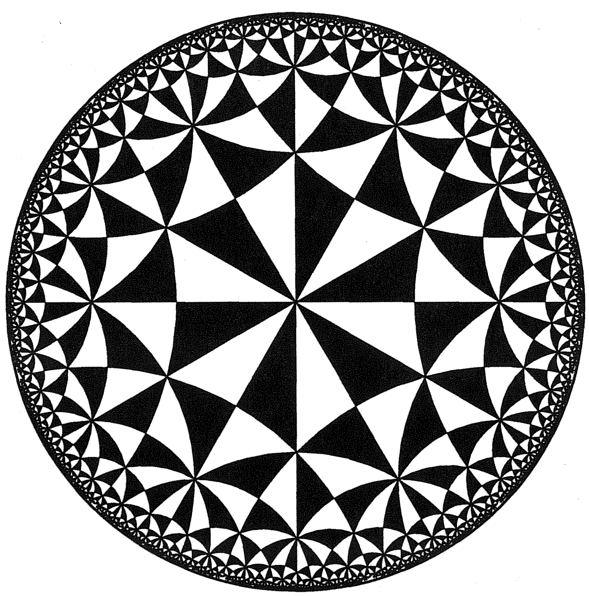

Figure 1: A pattern of triangles based on the $\{6, 4\}$ tessellation.

combine hyperbolic geometry and art in his four patterns *Circle Limit I*, *Circle Limit II*, *Circle*

Limit III, and *Circle Limit IV* — see Catalog Numbers 429, 432, 434 (and p. 97), and 436 (and p. 98) of [8]. The patterns of interlocking rings near the edge of his last woodcut *Snakes* (Catalog Number 448 of [8]) also exhibit hyperbolic symmetry. It is laborious to create repeating hyperbolic patterns by hand as Escher did. In the late 1970's, the first computer programs were written to create such patterns. Since then, much progress has been made in this area which spans mathematics, art, and computer science ([5] and [6]).

We will begin with a brief review of hyperbolic geometry and repeating patterns, followed by a discussion of regular tessellations, which form the basis for our hyperbolic patterns. Then, we will discuss symmetries, symmetry groups, and color symmetry. Finally, we will show more samples of patterns, and indicate directions of future work.

Hyperbolic Geometry and Repeating Patterns

By definition, (plane) hyperbolic geometry satisfies the negation of the Euclidean parallel axiom together with all the other axioms of (plane) Euclidean geometry. Consequently, hyperbolic geometry has the following parallel property: given a line ℓ and a point P not on that line, there is more than one line through P not meeting ℓ. Hyperbolic geometry is not very familiar to most people, and unlike the Euclidean plane and the sphere, the entire hyperbolic plane cannot be isometrically embedded in 3-dimensional Euclidean space. Therefore, any model of hyperbolic geometry in Euclidean 3-space must distort distance.

Escher used the *Poincaré circle model* of hyperbolic geometry which has two properties that are useful for artistic purposes: (1) it is conformal (i.e. the hyperbolic measure of an angle is equal to its Euclidean measure) — consequently a transformed object has roughly the same shape as the original, and (2) it lies entirely within a circle in the Euclidean plane — allowing an entire hyperbolic pattern to be displayed. In this model, "points" are the interior points of the *bounding circle* and "lines" are interior circular arcs perpendicular to the bounding circle, including diameters. The following are all examples of hyperbolic lines: the sides of the triangles in Figure 1, the sides of the hexagons of the $\{6,4\}$ tessellation in Figure 2, and the backbone lines of the fish in Figure 2.

However, the backbone lines in Escher's *Circle Limit III* pattern (see Figure 6) are *not* hyperbolic lines, but *equidistant curves* — circular arcs making an angle of approximately 80 degrees with the bounding circle (as explained by Coxeter [2]; each one is a constant hyperbolic distance from the hyperbolic line with the same endpoints on the bounding circle). Because distances must be distorted in any model, equal hyperbolic distances in the Poincaré model are represented by ever smaller Euclidean distances toward the edge of the bounding circle (which is an infinite hyperbolic distance from its center). All the motifs shown in the patterns in this paper are the same hyperbolic size, even thought they are represented by different Euclidean sizes.

A *repeating pattern* of the hyperbolic plane (or the Euclidean plane or the sphere) is a pattern made up of congruent copies of a basic subpattern or *motif*. All the following are examples of motifs: a triangle of Figure 1, a gray half-fish plus an adjacent white half-fish in the *Circle Limit I* pattern (Figure 2), and a hexagon in the tessellation $\{6,4\}$ (Figure 2). Also, we assume that repeating pattern fills up its respective plane. It is necessary that hyperbolic patterns repeat in order to show their true hyperbolic nature.

An important kind of repeating pattern is the *regular tessellation*, denoted $\{p,q\}$, of the hyperbolic plane by regular p-sided polygons, or *p-gons*, meeting q at a vertex. It is necessary that $(p-2)(q-2) > 4$ to obtain a hyperbolic tessellation; if $(p-2)(q-2) = 4$ or $(p-2)(q-2) < 4$, one obtains tessellations of the Euclidean plane or sphere respectively. The Euclidean plane, sphere, and hyperbolic plane are the three 2-dimensional "classical geometries" (of constant curvature).

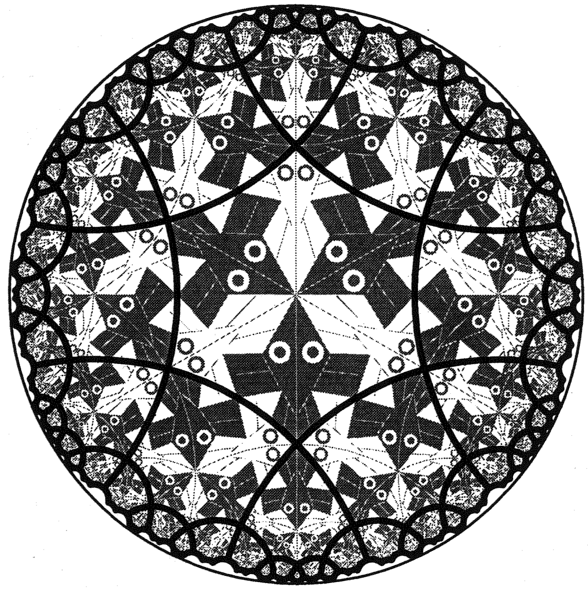

Figure 2: The $\{6,4\}$ tessellation superimposed on a computer-generated rendition of Escher's *Circle Limit I* pattern in gray and white.

Many of Escher's patterns are based on regular tessellations. Figure 2 shows the tessellation {6,4} superimposed on a computer-generated rendition of Escher's *Circle Limit I* pattern. Figure 6 shows the tessellation {8,3} superimposed on a computer-generated rendition of Escher's *Circle Limit III* pattern.

This completes our discussion of hyperbolic geometry, repeating patterns, and regular tessellations. Next, we consider the symmetry and coloring of patterns.

Symmetry Groups and Color Symmetry

Symmetric patterns are pleasing to the eye, so the patterns we consider have many symmetries — and as noted above, truly hyperbolic patterns must be repeating, and so must have some symmetry. A *symmetry operation* or simply a *symmetry* of a repeating pattern is an isometry (distance-preserving transformation) that transforms the pattern onto itself. For example, hyperbolic reflections across the fish backbones in Figure 2 are symmetries (reflections across hyperbolic lines of the Poincaré circle model are inversions in the circular arcs representing those lines — or ordinary Euclidean reflections across diameters). Other symmetries of Figure 2 include rotations by 180 degrees about the points where the trailing edges of fin-tips meet, and translations by four fish-lengths along backbone lines. In hyperbolic geometry, as in Euclidean geometry, a translation is the composition of successive reflections across two lines having a common perpendicular; the composition of reflections across two intersecting lines produces a rotation about the intersection point by twice the angle of intersection.

The *symmetry group* of a pattern is the set of all symmetries of the pattern. The symmetry group of the tessellation $\{p,q\}$ is denoted $[p,q]$ and can be generated by reflections across the sides of a right triangle with angles of $180/p$, and $180/q$ degrees; that is, all symmetries in the group $[p,q]$ may be obtained by successively applying a finite number of those three reflections. Such a right triangle is formed from a radius, an apothem, and half an edge of a p-gon. Those triangles corresponding to the the tessellation $\{6,4\}$ are shown in Figure 1, which thus has symmetry group

$[6, 4]$. The orientation-preserving subgroup of $[p, q]$, consisting of symmetries composed of an even number of reflections, is denoted $[p, q]^+$. Figure 3 shows a hyperbolic pattern with symmetry group $[5, 5]^+$ (ignoring color) which uses Escher's fish motif of Escher's Notebook Drawing Number 20 (p. 131 of [9]) and his carved sphere with fish (p. 244 of [9]); those patterns have symmetry groups $[4, 4]^+$ and $[3, 3]^+$ respectively.

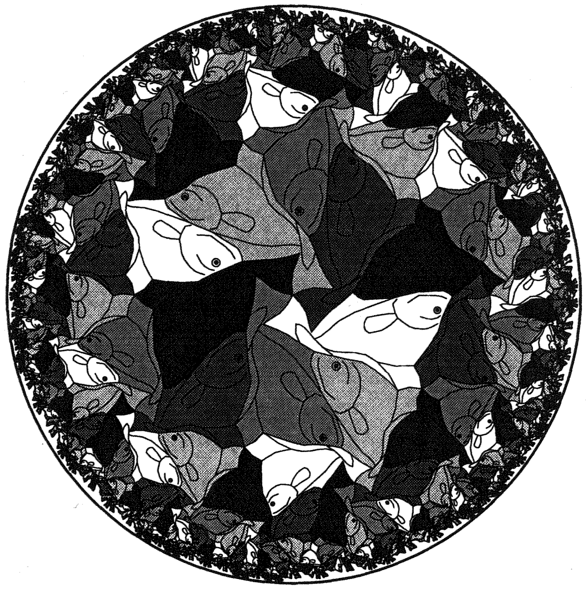

Figure 3: A hyperbolic pattern with symmetry group $[5, 5]^+$ using the fish motif of Escher's Notebook Drawing Number 20.

One other subgroup of $[p, q]$, denoted $[p^+, q]$, is generated by a p-fold rotation about the center of a p-gon and a reflection in one of its sides, where q must be even so that the reflections across p-gon sides match up. Figure 4 shows a pattern of 5-armed crosses with symmetry group $[3^+, 10]$ that is similar to Escher's *Circle Limit II* pattern of 4-armed crosses which has symmetry group $[3^+, 8]$. In these patterns, the 3-fold rotation centers are to the left and right of the ends of each cross arm, and $q/2$ reflection lines pass through the center of the crosses (and the center of the bounding circle). Escher made similar use of the group $[p^+, q]$ for his "angel and devil" patterns in Notebook Drawing Number 45, *Heaven and Hell* on a carved maple sphere, and *Circle Limit IV*, with symmetry groups $[4^+, 4]$, $[3^+, 4]$, and $[4^+, 6]$ respectively (see pages 150, 244, and 296 of [9]). H. S. M. Coxeter discusses these three patterns and their symmetry groups on pages 197–209 of [4]. Figure 5 shows a related pattern of devils with symmetry group $[5^+, 4]$. For more about the groups $[p, q]$ and their subgroups, see Sections 4.3 and 4.4 of [3].

The symmetric use of color can add to a pattern's aesthetic appeal. A pattern is said to have *n-color symmetry* if each of its motifs is drawn with one of n colors and each symmetry of the uncolored pattern maps all motifs of one color onto motifs of another (possibly the same) color; that is, each uncolored symmetry exactly permutes the n colors. This concept is often called *perfect color symmetry.* In all the examples, we disregard color when discussing symmetry groups. It is also important to adhere to the *map-coloring principle*: copies of the motif sharing a boundary segment must be different colors. We usually only say that a pattern has color symmetry if the number of colors is 2 or more.

Figures 1, 3, and 4 exhibit 2-, 5-, and 3-color symmetry, respectively. In contrast, note that the fish pattern of Figure 2 does not have color symmetry since the gray and white fish are not equivalent

Figure 4: A hyperbolic pattern of 5-armed crosses with symmetry group $[3^+, 10]$.

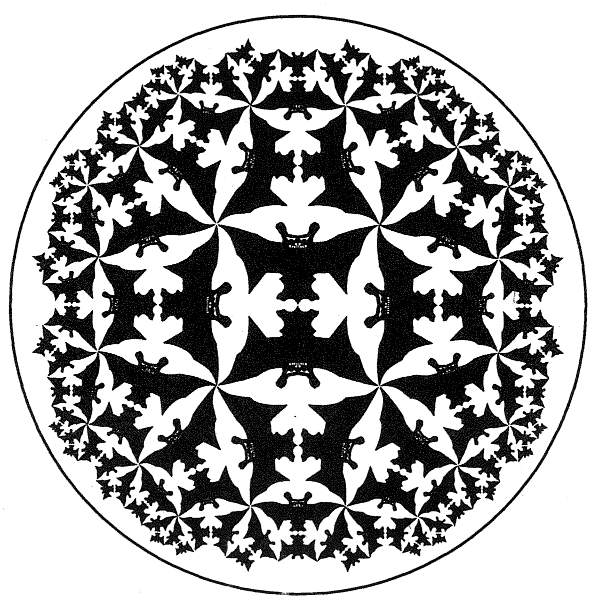

Figure 5: A hyperbolic pattern of devils with symmetry group $[5^+, 4]$.

(the gray fish have 60 degree noses and the white fish have 90 degree noses). Escher's print *Circle Limit III,* his most successful hyperbolic pattern, exhibits 4-color symmetry and is based on the tessellation $\{8,3\}$. In Figure 6 we show the $\{8,3\}$ superimposed on a computer-generated version of that pattern. The symmetry group of the *Circle Limit III* pattern is more complex than the examples we have seen so far — it is generated by three rotations: a 4-fold rotation about the right fin tip, a 3-fold rotation about the left fin tip, and a 3-fold rotation about the nose of a fish. The two different kinds of 3-fold points alternate around the vertices of an octagon of the $\{8,3\}$.

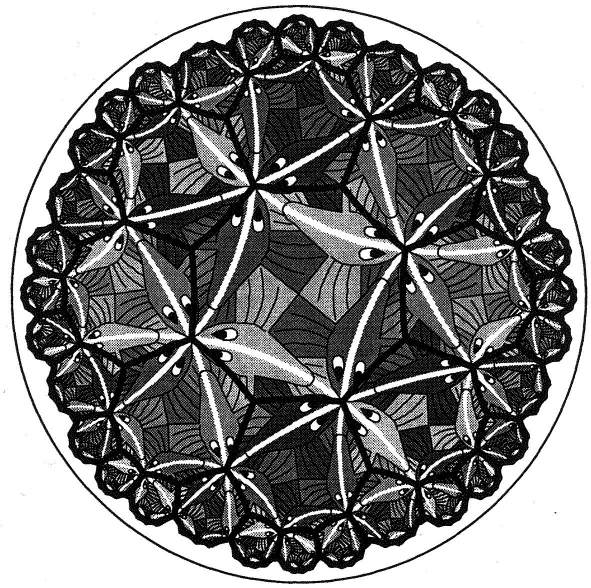

Figure 6: The $\{8,3\}$ tessellation superimposed on a computer-generated gray-scale rendition of Escher's *Circle Limit III* pattern.

Figure 7 shows a hyperbolic pattern with 6-color symmetry based on the tessellation $\{10,3\}$, using the fish motif of *Circle Limit III*. As with *Circle Limit III*, the backbone lines form equidistant curves, not hyperbolic lines. For more on color symmetry see [7], [11], and [12]. This completes our discussion of the theory of hyperbolic patterns. Next, we look at some examples.

Examples of Patterns

In 1958, the mathematician H. S. M. Coxeter sent Escher a reprint of an article that Coxeter had written [1]. In that article, his Figure 7 displayed the pattern of hyperbolic triangles based on $\{6,4\}$ that we have shown in our Figure 1. When Escher saw this pattern, it gave him "quite a shock" (Escher's words), since it solved his problem of showing a pattern "going to infinity" in a finite space. By examining Figures 1 and 2, it is easy to see how the triangles of Figure 1 could be modified to obtain the fish of Escher's *Circle Limit I* pattern. Thus, Coxeter's Figure 7 was the inspiration for Escher's *Circle Limit* patterns. In turn, those patterns motivated the author to design computer programs that could draw repeating hyperbolic designs.

Escher had two criticisms of *Circle Limit I*: (1) there is no "traffic flow" along the backbone lines — the fish alternate directions every two fish lengths, and (2) there are fish of both colors along each backbone line. He resolved these problems nicely in his *Circle Limit III* pattern. We show a different solution to the "traffic flow" problem in Figure 8, converting the *Circle Limit I* pattern to one based on the $\{6,6\}$ tessellation. The resulting pattern has 2-color symmetry, since

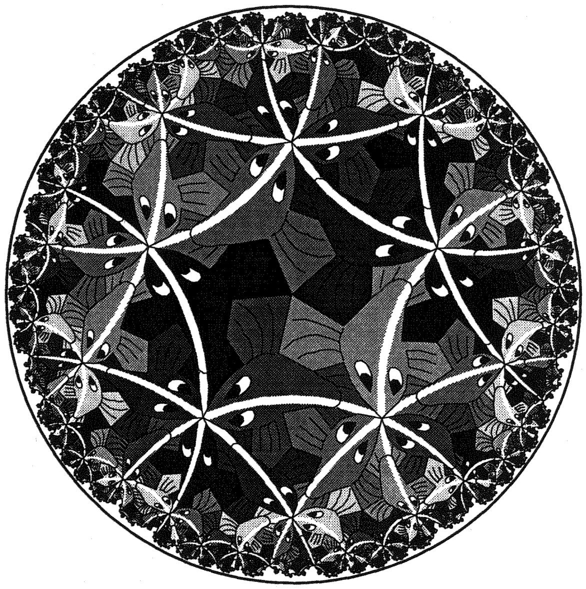

Figure 7: A 5-fish pattern based on the motif of *Circle Limit III.*

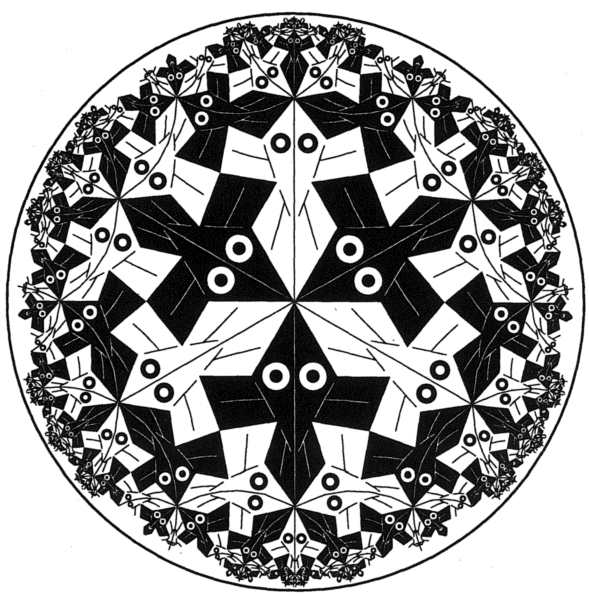

Figure 8: A hyperbolic pattern with 2-color symmetry, based on the $\{6, 6\}$ tessellation using the black and white fish motif of *Circle Limit I*.

black and white fish are interchanged by translating along backbone lines by one fish-length or rotating by 90 degrees about trailing fin-tips. We can solve Escher's second problem as well in Figure 9 by using three colors in the pattern of Figure 8, making the fish all the same color along a backbone line. The resulting pattern has 3-color symmetry.

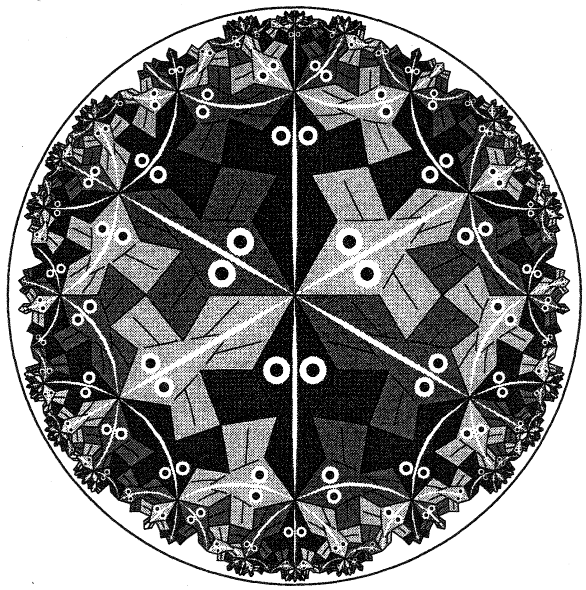

Figure 9: A hyperbolic fish pattern having 3-color symmetry and based on the pattern of Figure 8.

Escher carved a maple sphere with 12 lizards similar to those of his Notebook Drawing Number 25 (the sphere and Drawing 25 are shown on pages 245 and 135 of [9]). The pattern of Drawing 25 is incorporated in Escher's prints *Metamorphosis II and III* and *Reptiles* (pages 280, 326, and 284 of [8]). Figure 10 shows a related hyperbolic pattern. The carved sphere, Drawing 25, and Figure 10 are based on the tessellations $\{4,3\}$, $\{6,3\}$, and $\{8,3\}$ respectively. The left rear feet and left sides of the heads of the reptiles are centers of 3-fold rotations in all cases. The right rear knees meet at 2-, 3-, and 4-fold points respectively (of course the knees of the spherical lizards cannot meet at a 2-fold point — but the ankles do).

Escher did not create a spherical pattern based on Notebook Drawing Number 70 (p. 172 of [9]), but Schattschneider and Walker [10] did cover an icosahedron with this pattern (which could theoretically be blown up onto the circumscribing sphere). Figures 11 and 12 show related hyperbolic patterns with 7 and 8 butterflies respectively meeting at left front wing tips. Disregarding color, Drawing 70, and Figures 11 and 12 have symmetry groups $[6,3]^+$, $[7,3]^+$ and $[8,3]^+$ respectively. A pattern of butterflies meeting p at a left front wing tip and 3 at a right rear wing tip can be given color symmetry by using $p+1$ colors. However, if p is even, three colors are sufficient.

This completes our selection of sample hyperbolic patterns. In the final section, we indicate directions of future work.

Future Work

The current version of the computer program can draw repeating hyperbolic patterns with color symmetry whose symmetry group is a subgroup of $[p,q]$ and whose motif lies within a p-gon of the corresponding tessellation $\{p,q\}$. For more details on the pattern-drawing process, see [5] and [6]. It

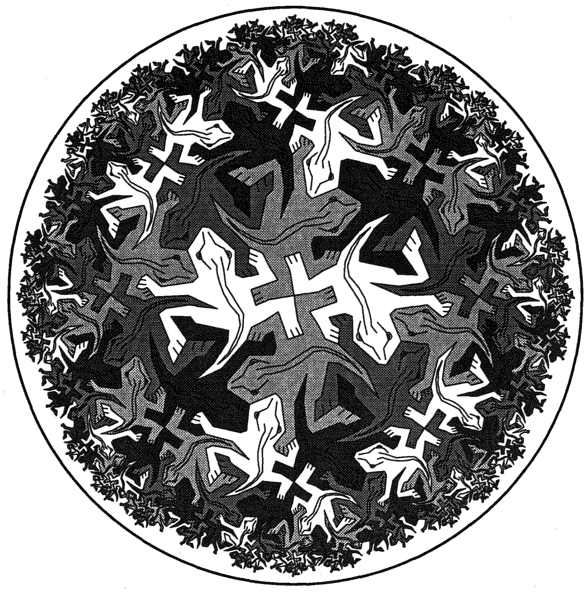

Figure 10: A hyperbolic pattern having 4-color symmetry, based on the $\{8,3\}$ tessellation and using the lizard motif of Notebook Drawing Number 25.

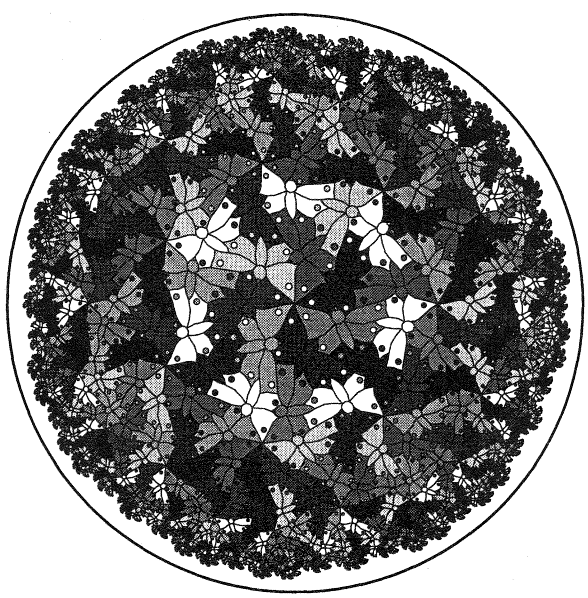

Figure 11: A hyperbolic pattern having 8-color symmetry, based on the $\{7,3\}$ tessellation and using the butterfly motif of Notebook Drawing Number 25.

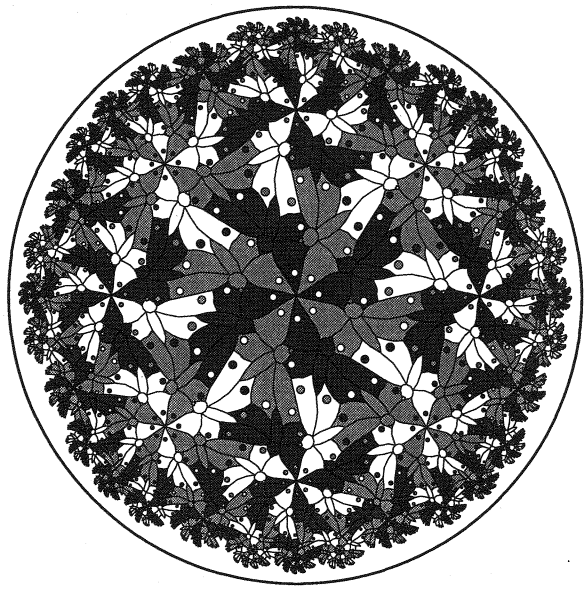

Figure 12: A hyperbolic pattern having 3-color symmetry, based on the $\{8, 3\}$ tessellation and using the butterfly motif of Notebook Drawing Number 25.

would be useful to be able to draw more complicated patterns than those based on $\{p, q\}$, including those like *Circle Limit III* and Notebook Drawings Number 25 which each have 3 kinds of rotation centers. At least two of the rotation periods had to be the same in the examples we have shown.

There are also programs to convert between any two hyperbolic motifs with different values of p and q, and to convert from a Euclidean motif to a hyperbolic motif. It would be be useful to be able to convert motifs between any of the three classical geometries: Euclidean, spherical, and hyperbolic. To this end, it would be useful to have a program to create repeating spherical patterns and print them out — onto polyhedra, for instance.

Finally, it would be useful to automate the specification of color symmetry of a pattern. Currently this must be done by hand. Considering Figures 11 and 12, which require 8 and 3 colors respectively, automatic color symmetry generation would seem difficult.

Thus there are many challenges left in creating artistic hyperbolic patterns.

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