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Mathematical Connections
in Art, Music, and Science

Music, Mathematics, and Magnetic Ordering

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Consider the problem of seating a dichotomous group (say, five scientists and three musicians) about a round table so as best to intermingle the two constituencies. There is an intuitive solution (up to rotation), at least for small numbers. The distribution of the seven months with 31 days among the 12 months of the year appeals to a similar intuition. Many Western musical scales and chords (considered as a selection from the 12 notes of the equal-tempered scale) reflect comparable distributions--called Maximally Even (ME) sets [1]. These musical patterns (and corresponding extra-musical patterns: seating arrangements, etc.) are well defined mathematically in terms of intervals between pairs of selected notes: specific intervals (measured by counting over the full 12-note scale) associate in a particular way with generic intervals (measured by counting only the selected notes). The definition of ME applies recursively to account for three-note chords (triads) drawn from the Western diatonic scale, as well as the seven-note gramas of ancient Indian music [2]. More generally, any set of notes (or extra-musical circular distribution) may be characterized in terms of its degree of evenness (minimal through maximal) through further interval analysis [3].

The physical connection is apparent in the application of the music theory formalism (mentioned above) to the interaction of Ising spins [4] on a one-dimensional lattice. The distribution of white and black keys on a piano can be visualized as a distribution of white and black dots on a line. Each white dot can represent an up spin-1/2 site on the lattice and each black dot can represent a down spin-1/2 site. In the Ising model the up (down) sites interact pairwise. In solid state physics (for large systems, i.e., the "keyboard" gets large) it is mathematically convenient to use periodic boundary conditions in which the "last" spin in the line interacts with the "first" spin and minimizes the energy of interaction of all (or a subset of) spins. By invoking periodic boundary conditions (referred to as "octave equivalence" in music theory) the mathematical theory developed to analyze music (Western as well as Eastern) can be extended to address the problem of interacting one-dimension Ising spins [5].

The confluence of music theory, mathematics, and physics leads to surprising connections among seemingly disjoint disciplines. Further generalization of the mathematics of music theory to the physics of magnetic ordering leads to interesting fractal magnetic phase diagrams [6] which has been discussed previously in the technical literature [7].

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