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Applications of Fractal Geometry to the Player Piano Music of Conlon Nancarrow

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Abstract

The relationship between music and geometry goes back thousands of years to the Greek quadrivium. Fractal structures have been explored in music and sound since at least 1978 (Gardner) and this work has recently been extended to specifically explore fractal structures in melodies (Mason and Saffle, 1994; Chesnut, 1996) and in musical forms and phrase structures (Solomon, 1998). Among the fractals that have been identified in musical structures are Sierpinski's triangle, Peano curves, and the Koch snowflake.

This paper is an effort to apply fractal observations to the player piano studies of American-Mexican composer Conlon Nancarrow. The most clearly mathematically-oriented of Nancarrow's Studies are the canons that explore mathematical relationships as simple as two voices in the relationship 3:4 or as complex as twelve voices proportional to the pitches of the justly-tuned chromatic scale. In particular, those of the canons which are also "acceleration canons"—that is, using carefully controlled rates of acceleration and deceleration among the voices—offer compelling possibilities for study of fractal properties. Among the studies which will be examined here are Nos. 14 (two voices) and 19 and 27 (three voices).

Introduction

The relationship between music and geometry goes back thousands of years to the Greek quadrivium. Fractal geometry is a relatively newly-described branch of mathematics based on the 1977 work of Benoît Mandelbrot [1] in which elements of self-iteration and scaling are recognized in a variety of naturally-occurring objects as widely diverse as coastlines and bodily structures such as the brain and bronchial lobes. Mandelbrot's theories began to be applied to music and sound beginning in 1978, with fractal structures being identified in the nature of sound itself [2, 3], in melodies [4, 5], and in musical forms and phrase structures [6].

Review of the Literature: Fractals, Music, and Sound

In studying fractal qualities in music, the properties of self-iteration, scaling, and space-filling have been the focus of study. According to Solomon [6], "Perhaps the most important defining property of fractals is self similarity on many different scales; i.e., they have self-iterating geometric structures that repeat in different sizes." Solomon uses the beautiful example of a fern frond, a natural object in which the same leafy shape is iterated on a number of different scales.

The *space-filling* property of fractals is also important in music. Consider Figure 1, a simple line fractal known as a Peano curve. With each new iteration of the generating shape, the space is more tightly filled and the length of the line drawing the curve increases. The number of iterations and the length of the line can reach infinity, moving toward filling the space but never completely doing so. In music, the property of space-filling takes place in the time dimension when a pattern is reiterated in proportionally shorter time values.

Let us now apply the same ideas of self-iteration, scaling, and space-filling to music. Figure 2 shows a short "generating motif" that can be compared to the largest triangle of a Koch snowflake. Notice how, with the second iteration of the melody, the same melodic shape is reiterated (self-iteration), in proportional time values (scaling), and more sonic space is filled. These are not new concepts to musicians, who recognize self-iteration in melodic imitation, scaling in rhythmic augmentation/ diminution, and space-filling (in a sonic sense) in the application of these procedures together.

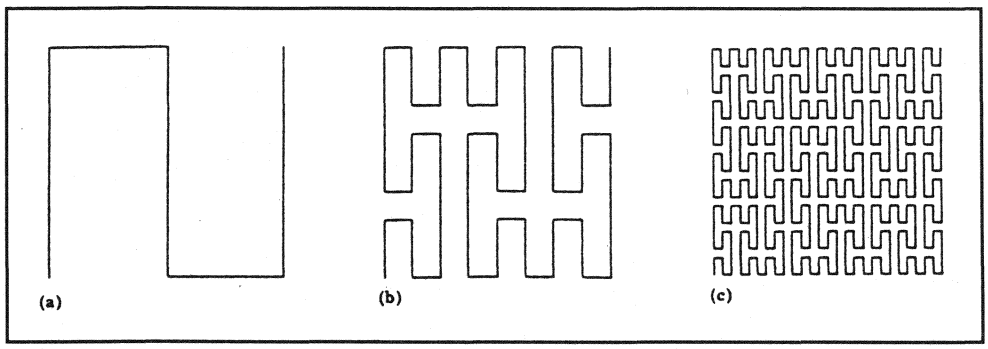
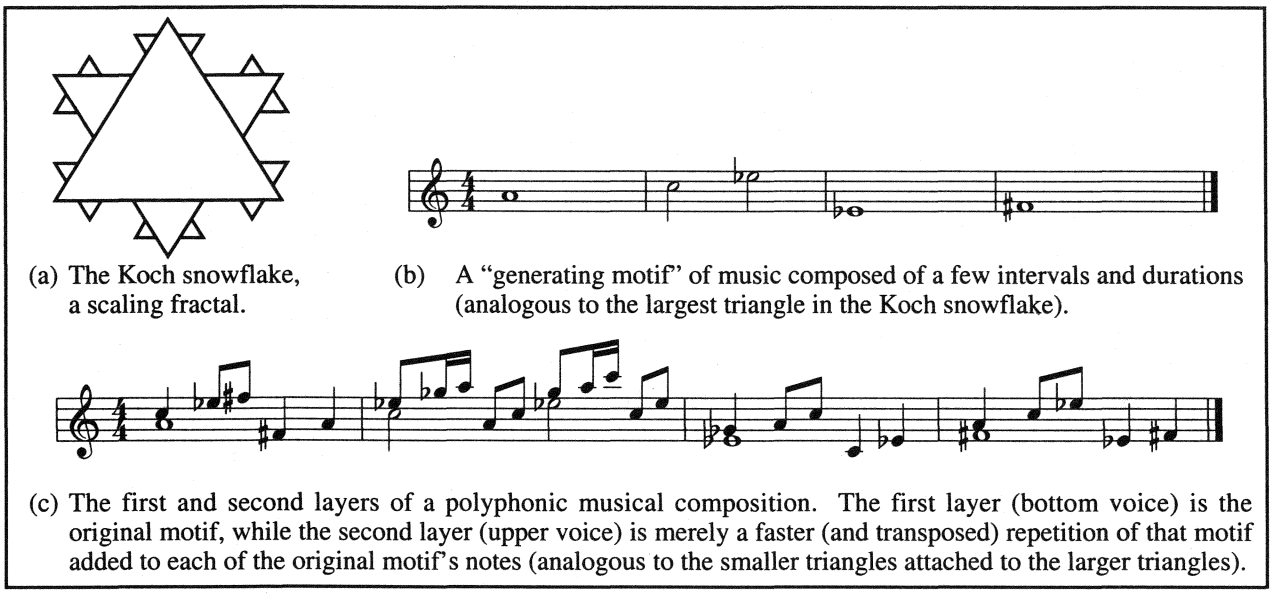


Figure 1: *A Peano curve ([4], p. 31), which illustrates the space-filling ability of fractals. With each further iteration of the curve, the length of the line drawing the curve approaches infinity.*



(a) The Koch snowflake, a scaling fractal.

(b) A "generating motif" of music composed of a few intervals and durations (analogous to the largest triangle in the Koch snowflake).

(c) The first and second layers of a polyphonic musical composition. The first layer (bottom voice) is the original motif, while the second layer (upper voice) is merely a faster (and transposed) repetition of that motif added to each of the original motif's notes (analogous to the smaller triangles attached to the larger triangles).

Figure 2: *Properties of self-iteration, scaling, and space-filling in a musical segment ([7], p. 190).*

Fractals can be observed in music in other ways. In 1978, Gardner [2] wrote of the work of Richard Voss, in which the nature of sound itself was revealed to have fractal properties. Mandelbrot and Voss discovered a special class of sounds in which the property of scaling is actually present in the waveform itself. These sounds, which Mandelbrot terms "scaling noises," have the fascinating property that the sound—including its pitch—does not change if the sound is played at a different speed.

Voss's work also focused on how the nature of sound relates to the construction of pleasing melodies. Gardner [2] describes how Voss identifies the frequency spectra for three types of "noise"—white, $1/f$ ("pink"), and Brownian—and demonstrates how the properties of these different waveforms could be interpreted as melodies. It turns out that $1/f$ ("pink") noise exhibits fractal self-similarity whereas white and Brownian noise do not, and it is the melodies based on $1/f$ noise that most people in a test audience found most appealing, based on the melodies' effective balance between complete randomness (surprise) and extreme correlation (expectation).

Fractal properties of melodic structures have been further studied by Mason and Saffle [4], who showed how right-angle drawings known as Lindenmayer (L-system) curves could be used to create melodies—albeit of questionable musical value. Melodies are created from the curves by interpreting horizontal line segments as durations and vertical line segments as pitches. Mason and Saffle also assert that many existing melodies can be shown to have strong correlations with L-system curves, although their work in 1996 is very preliminary. They did, however, identify L-system curves that "generate tunes

that are similar or even identical to hundreds of existing melodies by classical and popular composers" (p. 35).

Solomon [6] relates the ternary divisons and forms commonly found in music to the fractal known as Sierpinski's triangle (Figure 3), which he compares to a ternary (ABA) scherzo form. The ternary relation is obvious, but Solomon also notes that within the larger divisions one often finds binary divisions (such as a rounded binary form in the first A section), and these would relate to the binary division of the triangle's sides that create further iterations of the Sierpinski triangle.

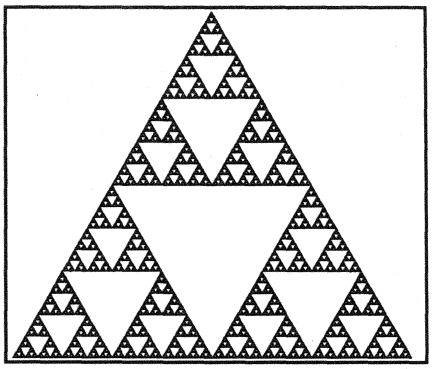


Figure 3: *An example of Sierpinski's triangle [6].*

And, of course, it should be mentioned here that the Schenkerian system of analysis has, as a primary goal, the identification of self-iterating melodic and harmonic patterns that are represented in both surface details and structural components—thus confirming both the properties of self-iteration and scaling as being inherent in many tonal compositions.

Finally, several writers have made fascinating speculations on a basic connection between fractal properties in nature, in the nature of sound itself, and our perception of musical beauty. Mason and Saffle's work [4] led them to conjecture about a fundamental relationship between our perception of beauty in melody and musical form and the presence of fractal qualities in music. As they state:

> Aspects of certain theories about the origins and fundamental structures of melodies suggest that much—perhaps all—beautiful music is, in some essential sense, fractal in its melodic material and internal self-similarity. (p. 35)

Later, they say:

> Is there something universally appealing about music—something that transcends individual cultures and tastes? We believe the human mind may use one or more models of perception in order to determine whether a given melody or musical structure is ugly or beautiful. (p. 36)

Gardner [2] notes that Mandelbrot, too, has raised similar questions in regard to abstract art:

> Is it possible, Mandelbrot asked himself many years ago, that even completely nonobjective art, when it is pleasing, reflects fractal patterns of nature? Mandelbrot has some unpublished speculations along these lines. He is fond of abstract art, and maintains that there is a sharp distinction between such art that has a fractal base and such art that does not, and that the former type is widely considered the more beautiful. (p. 24)

Nancarrow and His Player Piano Studies

Conlon Nancarrow began writing his remarkable player piano studies in the late 1940s, and by the time of his death in 1997 this body of work consisted of about 50 pieces. Nancarrow became interested in writing music based on "temporal dissonance," or multiple and often conflicting tempos, after reading Henry Cowell's book *New Musical Resources* in 1939. At this time Nancarrow was preparing to flee the United States for Mexico as a result of his Socialist party affiliations. He spent the rest of his life in Mexico,

working in virtual isolation until his music began to attract notice in his later years and he began to be regarded as an eccentric genius (confirmed in 1982 when Nancarrow was among the first class of recipients of the MacArthur Foundation's "genius" grants).

Nancarrow's interest in the player piano was a practical one: as a composer in the late 1930s, he was intrigued by Cowell's radical ideas about rhythm yet frustrated by the limitations of human performers in interpreting complex tempo relationships. He found the player piano to be the best means available at the time for realizing total control over performance. It is fortunate for us that Nancarrow reached this nexus in his compositional career when he did—he commented in later years that, had he not found the player piano, he would surely have turned in the 1950s to the electronic medium for his compositions.

More than two dozen of Nancarrow's player piano studies are canons, which are basically of two types. In the first type, what I will call "proportion canons," unchanging ratios of tempos are established among the canonic voices. These canons have ratios in their subtitles, for example "Canon 12/15/20" and "Canon $\sqrt{2}/2$." The second type of canon is the "acceleration canon," which uses carefully controlled rates of acceleration and deceleration among the voices to create what Gann [8] calls a "sense of curved time" (p. 146). These canons often have percentages in their subtitles, such as "Canon 5%/6%/8%/11%" (Study No. 27).

Study No. 14, "Canon 4/5" and Study No. 19, "Canon 12/15/20". We can begin our exploration of fractal properties in Nancarrow's studies with Studies No. 14 and 19, which are both part of a group of six canons (Nos. 14-19) that are based on the additive rhythm formula (n-1, n, n+1, n). In both canons, the total length of the canonic part (337 eighth note beats) is derived from three versions of the additive formula: 3+4+5+4, 5+6+7+6, and 6+7+8+7. The first pattern contains a total of 16 eighth notes, the second 24, and the third 28; the smallest common multiple of these patterns is 336, and to make the patterns converge at the end a final note is added for a total of 337 beats (see Figure 4). A fourth voice declaims the pattern 4+5+6+5, although this pattern is varied among the six pieces so that they do not all have the same rhythm. In Study No. 14 [10] the four patterns are collapsed into a resultant rhythm; the

Formula = **n-1, n, n+1, n**
Pattern 1: 3+4+5+4 = 16 eighth-note beats
Pattern 2: 5+6+7+6 = 24 eighth-note beats
Pattern 3: 6+7+8+7 = 28 eighth-note beats
Smallest common multiple = 336

Figure 4: *Derivation of additive rhythm formulas in Studies 14-19. A fourth pattern, 4+5+6+5, provides rhythmic variety among the six pieces. The total beat length of Studies No. 14 and 19 is 337 beats (i.e., beginning and ending with a convergence of the three patterns).*

resulting rhythmic attacks are shown in Figure 5. A glance at the rhythmic structure reveals another additive formula that occurs among the four voices: beginning with the top voice, additive patterns of 3+4+5+6, 4+5+6+7, and 5+6+7+8 (identified in the shaded boxes) can be traced descending toward the right, with each pattern lengthening in time. The patterns soon become obscured as they overlap, but before this happens I believe it is possible to perceive an arithmetical deceleration effect as the note values increase from the second measure ($\frac{4}{8}$) to the third ($\frac{6}{8}$). Whether they can be heard or not, the three additive patterns can be traced to the end of the piece, and represent self-similarity and scaling on a large scale.

Study No. 14, subtitled "Canon 4/5," is a two-voiced "proportional" canon in which the top voice states the canon 20% faster than the bottom voice. The bottom voice begins the canon, and the top voice enters at exactly the point that allows the two voices to converge in the center—at which point, the "follower" voice becomes the "leader" and vice versa. The interval of imitation is 2 octaves plus a fifth (see Figure 6). The bottom voice (♩=88) states the first 33.7 (337 x 20% = 67.4 / 2 = 33.7) beats of the canon before the top voice enters at ♩=110 (a tempo relationship of 4 to 5). Once the canon is underway, the top voice—going 20% faster—states 30 beats in each system to the bottom voice's 24 (in

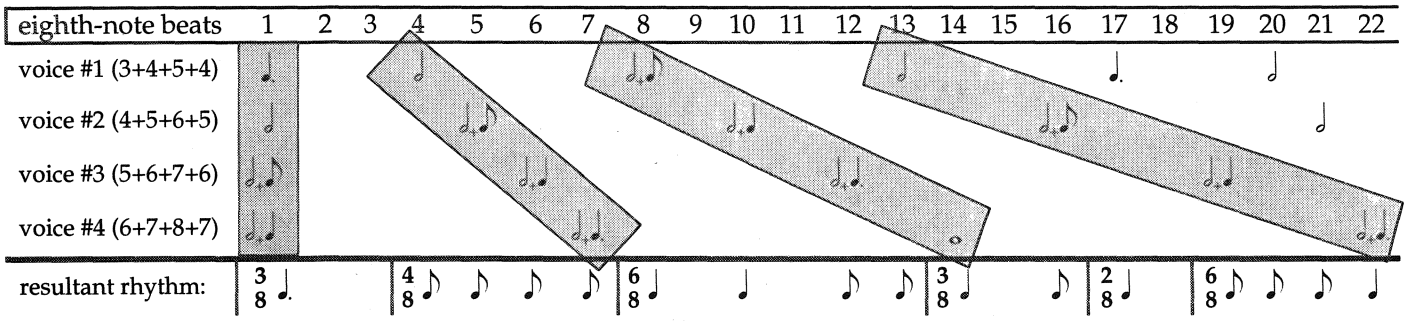


Figure 5: *Derivation of rhythmic attacks in Nancarrow's Study No. 14.*

Nancarrow's hand-written scores, the amount of space between notes is proportional to elapsed time). The two voices converge at the midpoint (beat 169), at which point the top voice becomes the leader; the piece concludes with the bottom voice stating its final 33.7 beats alone. Carlsen [9] calls this an "arch-shaped canon" (p. 18).

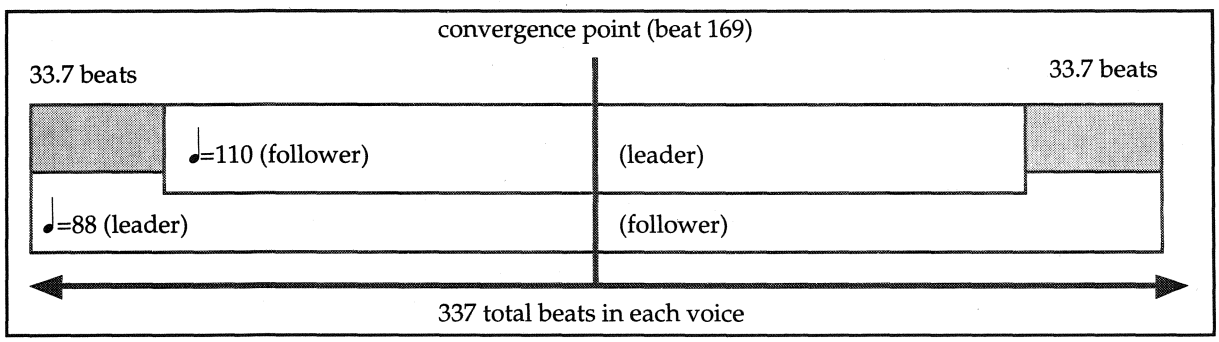


Figure 6: *Structure of Study No. 14. The top voice states the same musical material as the lower voice but at a higher pitch and a faster rate of speed. Canonic voices converge in the middle to form an arch.*

Study No. 19 [11], subtitled "Canon 12/15/20,"[1] uses three canonic voices and is constructed so that the point where all three voices converge is the very last note of the piece. In Study No. 19, the four additive rhythm formulas are clearly delineated into four distinct registers, opening with a 4-note chord that spans four octaves. Both Gann [8] and Carlsen [9] note that each of the four voices declaims basically the same melody, with the slower voices occasionally dropping a note in order to keep up with the faster voices. Carlsen [9] calls attention to this as a sort of Chinese-nested-boxes fractal relationship.

Like Study No. 14, No. 19 begins with the lowest (and slowest) voice (♩=144); it then adds a faster middle voice (♩=180) and a faster-still top voice (♩=240). In order for the convergence point to occur at the end, the first voice must state 67.4 eighth-note beats before the second voice enters. The interval of imitation between the voices is an eleventh, and the range of each voice is four octaves. Since the range of Nancarrow's piano is narrower than a standard piano by two keys on the bottom and three keys on the top, the entire keyboard is used, with the voices intentionally arranged so that the middle voice is symmetrical about the piano's middle note (E_4). The basic structure of Study No. 19 is shown in Figure 7.

[1] I can find no reason why Nancarrow specifically chooses the ratio 12:15:20 instead of the superparticular ratio 3:4:5 to which it reduces, despite the claim of Tenney [12] that 12:15:20 "incorporates" the three ratios 3:4, 4:5, and 3:5 used in this series (p. 51). This canon and No. 17 (which uses the same ratio, 12:15:20) are the only ones which use a non-reduced ratio in the subtitle.

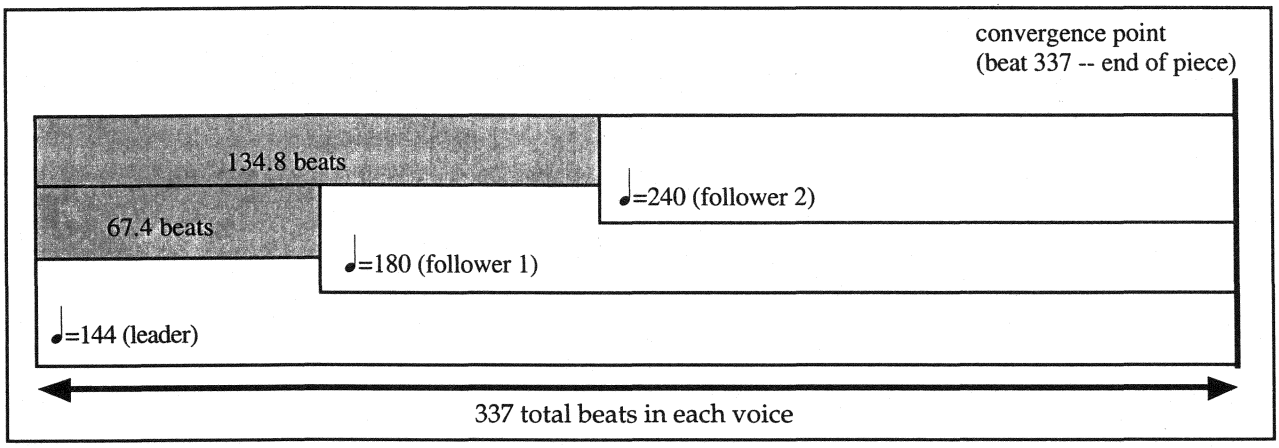


Figure 7: *Structure of Study No. 19. The musical material in the lowest voice is restated twice at progressively higher pitch levels and faster rates of speed. The voices converge on the very last note.*

Perhaps the best way to perceive the fractal nature of the overall forms of these two studies is by hearing them—the reader can listen to these brief pieces on [13] and [14]. The properties of self-iteration and scaling are convincingly represented in the simultaneous statement of identical (but transposed) material at different speeds—in each case, the first statement of the canon (the lowest voice) is the "largest" statement, to which are added successively "smaller" statements. Through the dimension of time, the space-filling aspect is convincingly portrayed by the progressively faster voices.

Study No. 27, "Canon 5%/6%/8%/11%." Let us look now at an example of an "acceleration canon": Study No. 27, described by Gann as "Nancarrow's acceleration tour-de-force" [15]. Nancarrow used two different types of acceleration/deceleration in his studies that had very different effects. The first type, arithmetical, is familiar to us already from the works of composers such as Messiaen. In arithmetical acceleration/deceleration, the same time value is subtracted from or added to each note to determine the length of the next note—for instance, a deceleration effect resulting from a sixteenth note to which is added on each successive iteration another sixteenth note, creating the pattern sixteenth note, eighth note, dotted eighth note, quarter note, etc. The resulting effect is not a smooth continuum, but a constantly increasing rate of change.

In geometric acceleration/deceleration, on the other hand, the rate of change is kept constant. The resulting rhythm is not a chain of standard note values and is too unwieldy to notate conventionally. Nancarrow found arithmetical acceleration to be adequate for small-scale effects, but geometric acceleration is far superior for the long, smooth acceleration and deceleration effects that could have structural significance. Study No. 27 [11] is one of eight studies Nancarrow wrote using geometric acceleration and deceleration; the percentages in the title indicate the four rates of acceleration and deceleration that are used in the piece. The piece also features a "clock" line in the middle of the texture that repeatedly states the same four pitches and forms a constant throughout the piece, creating a frame of reference against which the tempo changes can be heard.

Although in Studies No. 14 and 19 there is a clear mathematical basis for the rhythm of the voices, such does not appear to be the case for Study No. 27. Because there is no meter and the geometric acceleration technique requires spatial notation in the score, even in the clock line it would be difficult to discern a definite pattern. Whether or not any fractal structures will emerge in the rhythm is an area for further study.

Unlike the simpler forms of Studies No. 14 and 19, Study No. 27 is actually constructed of a series of 11 different canons. Its texture is also more complex because of the greater number of voices: there are four canonic voices plus the clock line in the middle. Within this structure, however, are smaller structures that are similar to the simpler canons in their fractal nature. Near the end of the piece, the structure of the ninth canon overlaps the four voices so that they are symmetrical about their centers in the same way that Study No. 14 was (see Figure 8). The voices progressively enter from highest to lowest in

descending 1/2-step increments: the first voice on D, the next on C#, then C, and finally B.[2] This chromatic progression happens to be a transposed version of the four chromatically adjacent pitches which comprise the clock line, whose notes are D#, E, F, and G♭.

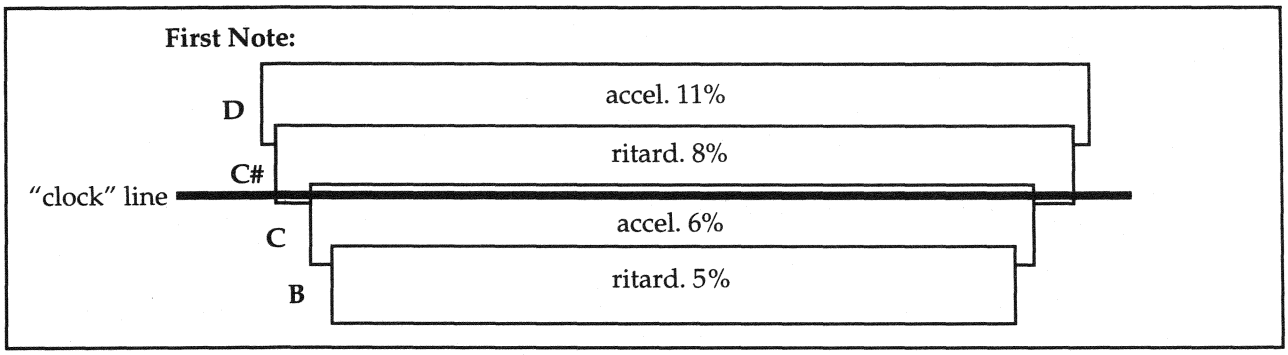


Figure 8: *Diagram of the structure of the ninth canon in Study No. 27. Each voice states the same musical material at a different pitch level and at a different rate of acceleration or deceleration (ritard.=ritardando).*

In this piece, the coincidences of the numerous canonic structures can be quite striking to the ear when they emerge from the contrapuntal texture. The very end of the piece is an excellent example—as in Study No. 19, Nancarrow sets up the four voices so that they converge on the last note in an audible fractal structure, with the same melodic pattern in all the voices but at different scales (rates of speed). On the final ascending scalar pattern from A_3 to G_4, each voice is assigned a rate of acceleration that becomes progressively faster in relation to the registral placement of the voice: the lowest voice accelerates at 5%, the next highest at 6%, the next highest at 8%, and the highest voice at 11%. The smoothness of the geometric acceleration as the voices race progressively faster to the final note is a stunning effect.

The reader is encouraged to hear the piece in its entirety on [14].

Conclusion

Nancarrow's interest in temporal relationships, realized through the structure of the canon and the medium of the player piano, provided him the means to create what we can now recognize as fractals in sound. The pieces discussed here represent only a small portion of Nancarrow's ingenuity in manipulating musical resources such as tempo and form to convey the fractal qualities of self-iteration, scaling, and space-filling. The space-filling quality, in particular, finds new expression in Nancarrow's canons through the dimension of time as expressed in tempo.

It is my hope that this brief outline of the study of fractals in music over the past twenty-plus years, and the overview of Nancarrow's canons and possible fractal applications for their study, will reveal exciting new possibilities for the further study of both. Scholarship in both areas is still emerging and dates back only a few years. There is undoubtedly a great deal more to be revealed on the fractal nature of music, and the player piano music of Conlon Nancarrow offers a unique and exciting field of study for identifying fractal structures in music.

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[2]There is an error in the score at the end of this canon. The second voice from the top (ritard. 8%) should end on a G octave in order for the voices to end chromatically on F, G♭, G, and A♭. Instead, the final two octave chords in this voice are in error, stating E and A instead of the required D and G.

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