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Mathematical Connections
in Art, Music, and Science

The Pi Symphony: Numeric Sequences in Music

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Abstract

The Pi Symphony, an original orchestral work, makes use of various methods of conveying a numeric sequence in music. I will describe these methods and show examples. Using the digits of pi and 'e' as melodic and rhythmic elements is the impetus of the Pi Symphony. Although it is commonly known that math and music are closely related and that there are many levels of complexity at which they may be linked, the Pi Symphony employs simple methods of linking these two subjects such as the mapping of a sequences of digits to a melodies, to harmonic structure and to rhythms.

The second movement of the Pi Symphony makes extensive use of the circle of fifths, to compliment the theme of pi, the ratio of a circles circumference to its diameter. Music, being an artistic endeavor, can inspire and be inspired by logic and emotions. The inspiration for the Pi Symphony comes largely from the moment of academic discovery, and the actual melody of pi itself.

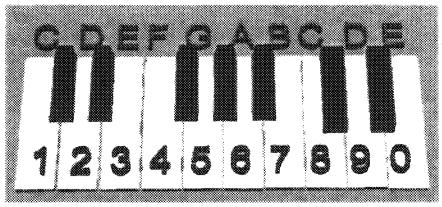
Concept and History

The original inspiration for the Pi Symphony arose from a habit of making numbers into melodies (e.g. phone numbers, zip codes, or any sequence of digits). While sitting in a calculus class in 1990, I mapped the first few digits of pi, around ten or so, and was intrigued by the fact that the melody was quite musical. When I looked up more digits, I found that the melody is quite pleasing for 32 digits! At this point in the sequence, the first zero occurs. Using a MIDI (Musical Instrument Digital Interface) sequencing program, I 'played' several hundred digits beyond that and found no other 'non-random' sounding melodies from which to draw.

The Mapping Algorithm

The digits are mapped to a melody by the simplest of methods. In any key, the major scale tones are numbered from 1 to 7, with 1 denoting the root, and continuing up to the octave at 8. One note above the octave is numbered 9 and two notes above the octave, the "tenth" interval, is indicated by the digit 0.

In the diagram below, the key of C is used to demonstrate a major scale mapping.



Using this method, 3.14159265358979323846264338327950... (in the key of C)
becomes: E CFCGDDAGEGCDBDEDDCFADAFEECEDBDGE...

Another example, 'e': 2.7182818284590452353...
becomes: D BCCDCCCDCFGDEFGDEGE (also in C)

By using this range, which covers an octave and a third, a reasonable number of notes are available to the melody. With only three notes represented twice, the entire octave is covered with minimal overlapping. Another benefit of mapping to a scale, as opposed to a chromatic mapping, is that different modal variations are then available with the same sequence of digits, e.g. major and minor.

How pi is used in the score

Most prevalently, pi is used as a melodic element: forwards, occasionally backwards, and even upside down, and once upside down and backwards. In the opening notes, pi is rounded off to 3.1416. Very soon, more digits are introduced, and in the first section after the opening, the first 32 digits of pi are played melodically.

'e' the base of the natural logarithms is also used in a straightforward manner melodically. Another tiny melody is derived from Planck's constant. Max Planck's constant, 6.626×10^{-34} (Joules-seconds), is integral to the study of quantum mechanics.

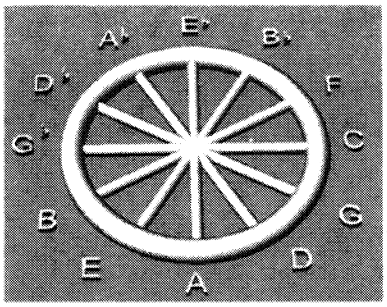
Rhythmically, the digits of pi are also used in a sort of 'pulse dialing' type of method. For example, a single note might be played three times with a rest, one time with a rest, four times with a rest, etc. for 3.14.

To play 'pi' on a snare drum, for example, the sequence would look like this:

```
xxx x  xxxx  x  xxxxx  xxxxxxxxx xx  xxxxxx xxxxx xxx  xxxxx  xxxxxxxx
  3 .1   4   1   5       9       2    6     5    3     5      8     ....
```

I toyed with the idea of representing each note rhythmically as dictated by the exponentially smaller nature of the values of the digits, but this was not musically feasible, since, after just a few digits there would have been values that became exceedingly small. Instead, I chose to make each note more-or-less equal valued with respect to time.

Later in the score, pi is played in a minor key, and even "transformed" to fit over the "circle of fifths".



The Circle of Fifths

In music, there is a well-known sequence of notes called the circle of fifths. Essentially, if one starts at any note and plays a perfect fifth above it, a perfect fifth above that, and continues on, all twelve tones of the scale would be played and the sequence would then repeat.

For example, the twelve tones of the western scale, played chromatically are as follows:

C, C#, D, D#, E, F, F#, G, G#, A, A#, and B

Using flats, instead of sharps, the sequence is:

C, Db, D, Eb, E, F, Gb, G, Ab, A, Bb, and B

When played through the circle of fifths, the sequence, starting on C would be this:

C, G, D, A, E, B, Gb, Db, Ab, Eb, Bb, F, and C again...

Of course, if you keep going up a perfect fifth, then after going once through the circle you end up in a quite high register. To stay in the same register, or 'range of pitch', one can go up a fourth, and down a fifth, up a fourth, and down a fifth.

To play through the circle of fifths backwards, you simply advance a perfect fourth interval each time.

Western music frequently uses this sequence of movement by fifths, and there are some interesting ways in which a chromatic sequence interacts harmonically with the "circle" sequence played simultaneously. Most 'songs' whose harmonic roots can be traced to Bach, or even earlier, use a small portion of the circle of fifths to define the harmonic structure. For example, the 'ABC' song, or 'Twinkle Twinkle Little Star' has a chord structure (in the key of C)

C F C G7 C G7 C C G7 C G7 C G7 C G7 C G7 C F C G7 C G7 C

The above sequence uses three nodes of the circle of fifths.

An excerpt from J.S.Bach's Two-Part Invention #4 reveals this harmonic progression:

Dm A7-9 Dm A7 Dm A7-9 Dm Gm9 C F Bb Em^{-5} Am Dm Gm C7 F ...

Notice that first six chords toggle between two nodes of the circle. Then the 6th thru the 11th chords in the sequence follow directly around the circle. The twelfth chord, an Em^{-5} jumps across the circle and the sequence continues in the same direction on the circle for six more chords.

Jazz music, especially, makes use of dissonant chords derived from the harmonic extensions of a tone center, and nearly every jazz standard has a portion of chord structure that can easily be mapped onto the circle.

The way that I have used the circle in the second movement of the Pi Symphony is to go around it completely for the harmonic structure several times with the melodies of pi, 'e', and even Planck's constant incorporated. It is, admittedly, a symbolic relationship, but this one entity, the circle of fifths, is easily the most prominent element of music theory, much like pi permeates mathematics.

Other ways of mapping sequences: Harmonic structure and rhythms

Not only is pi used melodically and rhythmically, but it is also used to create the harmonic structure (the chord changes) within some of the Pi Symphony. By playing out the chords based on the scale tones of the melody of pi, one can overlay the melody of pi onto the changes of pi for a pleasing effect.

For example:

```
Melody      ===> 3.14159265358979323846264338327950
Chords      ===> I     IV  I   V   ii  ii  vi  V
Chord Digits ==> 1      4  1   5   9    2  6   5
```

In this case 9 and 2 map to the same chord an octave apart, as do the 8 and 1 and 0 and 3 respectively.

Also, the melody of 'e' (e=2.7182818284590452353...) can also be mapped onto itself and it harmonizes quite well.

By playing the melody of 'e' as a sort of fanfare, and breaking it up into two-note segments, the sequence of digits can be used as a melody and as a harmonization to itself. Like this:

```
 DIGITS of e :     2.7   18 281828     4 5  90 452353...
 DIGITS of e :                          2.7  18 281828...

First Melody:      DB    CC DCCCDC     FG   DE FGDEGE
 2nd  Melody:                          DB   CC DCCCDC
```

The Spirit of the Pi Symphony

In my life, I have been inspired by the logic of musical sequences at a very early stage. I spent years avidly searching out the logical connections of chords, melody, and sequences, and how they related to one another. I derived much joy from this knowledge --as it appeared to be a window into a perfect world, a glimpse into the logic of the universe.

Later, when I studied physics, I realized that the logical interactions of math, the calculus, and engineering offer a similar yet vastly more complex view of the nature of the physical universe. The joy that I felt while discovering, for myself, new concepts (to me) in my studies reminded me of feelings I had while unravelling the logic of music. I wondered at the extreme elation that researchers and mathematicians and scientists must have when being able to expand our collective knowledge.

The emotion of the Pi Symphony is that elation. A calculus professor once quipped, " e^i(pi) +1 = 0 , was a meeting of the 'big five' in math, e, i, pi, 1, and zero." The melodies of pi and 'e' are used extensively throughout both movements of the symphony. In the second movement, much of the harmonic motion is straight through the circle of fifths with pi and 'e' interspersed triumphantly throughout.

The opening five notes of the Pi Symphony, and the closing five notes, are pi rounded to four places, 3.1416. They symbolically open and close a door on the endless sequence of transcendental digits of pi.

The State of the Score

The score for the Pi Symphony is in the 'Finale' music scoring software format. It is orchestrated for a full orchestra with woodwinds, brass, percussion, full strings, and harp. The version that will be played is a synthesized version using digitally sampled instruments, the notes of which are generated by the score itself via MIDI (Musical Instrument Digital Interface).

One goal is to debut the Pi Symphony with real instruments. Donations are being solicited via the internet (http://www.pisymphony.com) to be used toward that end. As of this writing, the score is going through a final edit stage, and parts are being generated for an initial 'read-thru' by a non-professional orchestra. After that is accomplished, I will pursue a performance and simultaneously assemble a digitally recorded performance by hiring individual players to record parts with a recording of the rest of the group.