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Hidden Geometry in Music of Bach and Schoenberg: Reflection, Rotation, Proportion

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Abstract

Musicians often equate the composers Bach (1685-1750) and Schoenberg (1874-1951) with a nearly obsessive relationship to numbers and mathematics. We think often of Schoenberg as a father of the numerical music organization systems of atonality and dodecaphony while there is an oft cited encoding within Bach's works the numerical and musical representation of his surname. Analysts often limit these composers' creativity to numerical surface details in the case of Bach or the tabulations of 12-tone row forms in the case of Schoenberg; this paper illuminates elegant architectural structures so prevalent behind such musical edifices. It comprises an architectural, geometrical, and statistical walking tour of two beautiful constructions, namely Schoenberg's Op. 19 No. 2 for piano and the Courante from Bach's Suite for Solo Cello No. 2. Through graphic measurements taken through modeling each piece of music onto a pitch-time two-dimensional complex plane (after Cogan and Escot), the paper analyzes internal architectural and geometric proportions of these works illuminating a consistent use of arithmetic, geometric, harmonic, and golden mean proportions amid these composers' works. It also provides graphical illumination of various pitch and time bilaterally symmetrical structures within the Schoenberg. Statistical contour correlations and oppositions are also found between ordered pitch data sets obtained from equal-length sections of the Bach using Spearman, Pearson, and Kendall data correlation methods. Finally, the paper compares the statistical distribution of pitch classes within the Schoenberg underlining his use of statistically lowest total duration pitches as contextually unique information at architecturally significant moments.

1. Bach

1.1 Architectural Proportions. The Courante from Bach's second suite (BWV 1008) for violoncello composed in 1721 exhibits a number of remarkable proportions. Using a two-dimensional pitch-time complex plane (Figure 1) in which each unit along the abscissa equals one quarter note of time duration and each vertical unit equals one semitone from C2 to G4 the movement exhibits the following proportions:

- Sections A and B are the same length – 48 quarter note durations.
- The first iteration of the nadir pitch C2 at time point 117 is at the golden section of the movement.
- Pitch collection A2, Bb2, D3 of section A – a local nadir - is sounded at the negative golden mean of section A.
- The Apex of section A is at its golden mean.
- Pitch Bb3 at time point 24 of section A is at the golden mean of local apexes at time points 14 and 30; the same Bb3 also bisects section A.
- Pitch C2 at the nadir of section B is at the geometric mean at time point 69 (time points in this case counted without the repeats) of time points 48 and 96 quarter notes.
    - $(\sqrt{48 \times 96} \approx 69)$
- Pitch E2 at the nadir of section A is at the geometric mean at time point 69 (repeat included) of time points 48 and 96.
    - Notice that the geometric mean proportions meet at the midpoint of the piece.

- Local apex D4 at time point 15 of section A bisects the onset and apex (time point 30) of section A.
- Pitch G4 at the apex (time point 77) and pitch C2 the nadir (time point 117) are equidistant from the center of the movement.
- Pitch G4 at the apex of section B is at the reflection over the midpoint of section B of the harmonic mean. (i.e. Negative harmonic mean)
    - $\frac{2\times 48\times 96}{48+96}=64$, 64 reflected over the midpoint (72) of section B = 80
- The onset of the repeat of section B is at time point 144, a Fibonacci number.
- Pitch D4 at time point 66 is at the negative golden mean of section B.
- Pitch E4 at time point 77 is at the golden mean of section B.

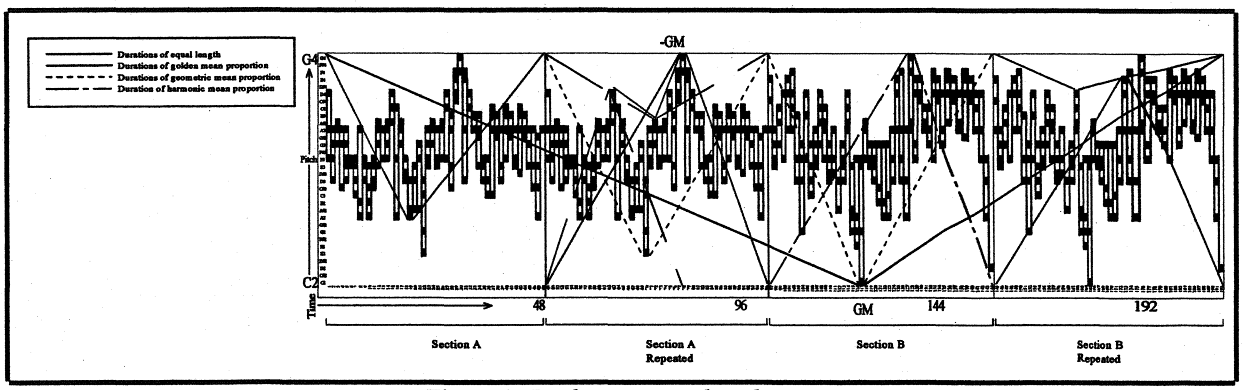

Figure 1: *Pitch-time complex plane*

Figure 2: *Courante – Bach suite for solo violoncello No. 2*

Notice from the above that sections A and B are of equal length and repeated. The graph of Figure 1 shows the actual physical and temporal development as the piece is performed, including repeats. Notice also from both Figures 1 and 2 that each half (sections A and B) seems to exhibit similar contour.

1.2 Pitch Data Correlations. When the above graph is converted into a data set in which each available pitch is given a unique integer (i.e. the bottom pitch on the graph is given a one, the next higher a two, etc.) and the collection of pitches is ordered from the first to the last such that each equal time duration is given its corresponding pitch integer, common statistical data correlation methods can determine similarities of pitch contour between equal-length sections of the construction. When the above graph is similarly converted into a data set in which, rather than mapping the collection of pitches in register, pitch-classes are mapped (i.e. each C-natural is given a unique integer, etc.), the same statistical data correlation methods can determine similarities of pitch-class content between sections A and B (Figure 3). The notion of pitch-class stems from the assumption that any group of pitches at octave-divisible distances, all A-naturals for example, sound equivalent to a large degree regardless of register.

■ Explanation of Statistical Correlation Methods

Pearson's correlation coefficient between x and y, $\sum_i (x_i-\bar{x})(y_i-\bar{y}) / \left(\sqrt{\Sigma_i (x_i-\bar{x})^2}\,\sqrt{\Sigma_i (y_i-\bar{y})^2}\right)$

Spearman's rank correlation coefficient between x and y, $((n^3-n)/6-T_x-T_y-\Sigma_i r_i^2)/\sqrt{((n^3-n)/6-2\,T_x)\,((n^3-n)/6-2\,T_y)}$ where n is the length of the lists, r_i is the rank difference between x_i and y_i, T_x is the correction term for ties in *xlist*, and T_y is the correction term for ties in *ylist*

Kendall's rank correlation coefficient between x and y, $(n_c-n_d)/\sqrt{(n_c+n_d+n_x)\,(n_c+n_d+n_y)}$ where n_c is the number of concordant pairs of observations, n_d is the number of discordant pairs, n_x is the number of ties involving only the x variable, and n_y is the number of ties involving only the y variable

(Data set not including the reduntant repeats - see bar graphs)

Pitch Correlations:	Pitch-Class Correlations:
Correlations between 1st and 2nd 50% (Sections A&B)	Correlations between 1st and 2nd 50% (Sections A&B)
0.706549 (Pearson)	0.761763
0.764576 (Spearman (Rank))	0.824566
0.609413 (Kendall (Rank))	0.64623
Correlations between 1st and 2nd 25%	Correlations between 1st and 2nd 25%
0.530207	0.451958
0.475785	0.445614
0.410891	0.338462
Correlations between 1st and 3rd 25%	Correlations between 1st and 3rd 25%
0.698515	0.649881
0.715776	0.789411
0.589764	0.651122
Correlations between 1st and 4th 25%	Correlations between 1st and 4th 25%
0.596909	0.736621
0.402708	0.740751
0.329711	0.562569
Correlations between 2nd and 3rd 25%	Correlations between 2nd and 3rd 25%
0.470109	0.355555
0.57517	0.550464
0.455834	0.412907
Correlations between 2nd and 4th 25%	Correlations between 2nd and 4th 25%
0.481557	0.622901
0.491387	0.641984
0.402321	0.531315
Correlations between 3rd and 4th 25%	Correlations between 3rd and 4th 50%
0.647788	0.693746
0.477429	0.693955
0.396847	0.548458

Figure 3: *Data Correlations*

Observe that there exists great statistical pitch contour similarity between sections A and B as noted when examining the complex plane graph (Figure 1) above. Note also that contour similarity diminishes when comparing subsections by further halving sections A and B to obtain divisions of 25% of the total duration of the piece. The above outlines also striking internal oppositions of pitch and pitch-class structure. Namely, the first halves of sections A and B (labeled 1st and 3rd 25% above) share great similarity in both pitch and pitch class domains whereas halves within sections (1st compared to 2nd 25% and 3rd compared to 4th 25%) share considerably less contour and pitch-class information. Also note that the correlation between the 2nd and 3rd 25% (spanning the border between sections A and B) exhibits the very least similarity among adjacent divisions in both contour and pitch-class domains. This recursive similar/dissimilar structure begins to illuminate an interesting phenomenon found in many "AABB" pieces of Bach [1]. Most listeners will experience the borders between repeats of sections A and B and between sections A and B themselves particularly jarringly due to almost complete change of musical context at those points as illustrated by the correlations above. Bach places the first 25% of the piece (first 50% of section A) squarely in the key of D minor and gradually modulates to the dominant key of A major (and for a short while, A minor) within the second 25%. The listener is then jolted suddenly back to the beginning of the piece (D minor) hearing section A again. The same process is repeated within section B beginning with A major, gradually modulating to D minor and suddenly, once again, jerking the listener back to A major, this time with a gigantic leap of more than two octaves. We can see from bar graphs (Figure 4) below the magnitude of oppositions of pitch and pitch-class usage between sections. Notice also that the first 25% and the final 25% exhibit great statistical similarity, indicating a return to opening materials (D minor key).

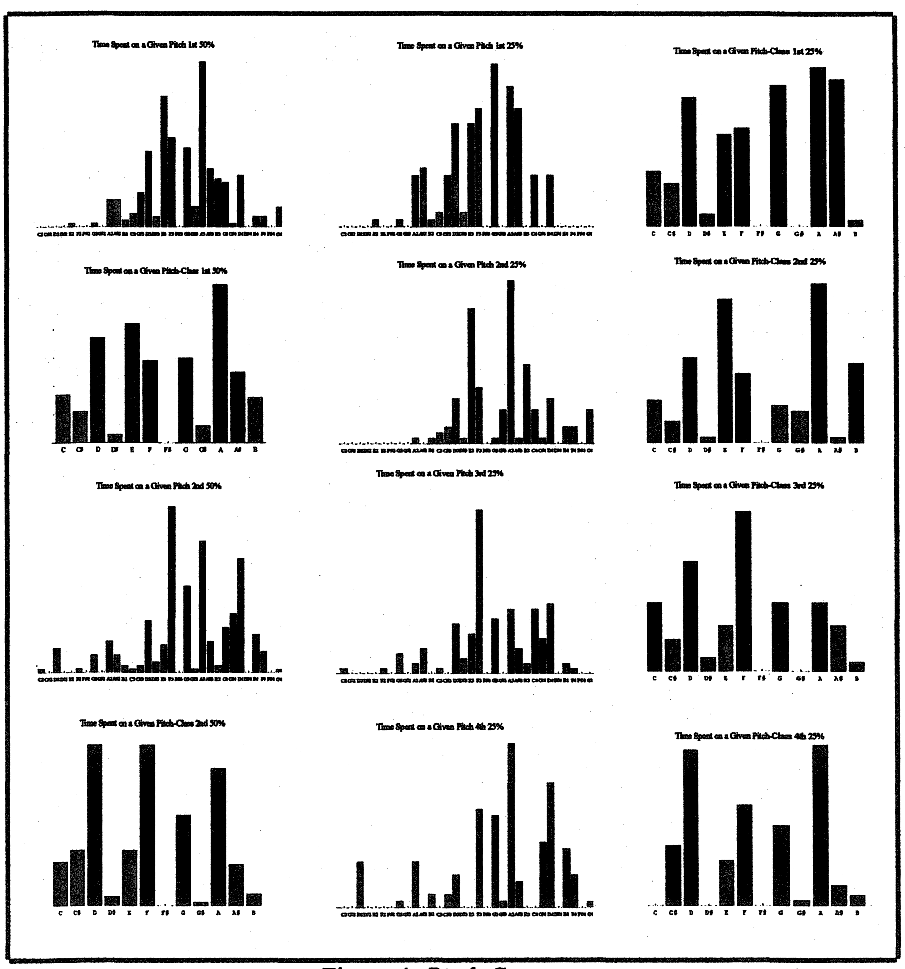

Figure 4: *Pitch Content*

2.2 Architecture. Schoenberg's Opus 19 No. 2 for solo piano composed in 1911 (nearly two hundred years after the Bach piece) exhibits remarkably similar architectural proportions and processes. Using a two-dimensional pitch-time complex plane as above, the music exhibits the following proportions:

- Range (semitones) represents approximate golden mean (GS) proportion relative to time (eighth note). Ratio of the length of X Axis in durational values to the length of the Y Axis in semitones = Golden mean -- 46 semitones / 72 eight notes)
- Significant events structurally are represented in similar relative positions in the pitch and time domains (+GS and –GS, .44 and .54 – lines represent the same positions relative to minima and maxima on both the pitch and time axes).
- Golden Sections (+,-) in both pitch and time domains mark significant events.
- When the 14 Eighth note introduction is subtracted from the piece (See dark lines):
    - o The remaining 57 eighth notes are bisected by the positive golden mean of the piece.
    - o The golden mean of the remaining 57 eighth notes is found at the end of the chord containing the lowest pitch of the piece. (Notice that the golden section (labeled GM) of the piece as a whole is found at the beginning of this chord.)

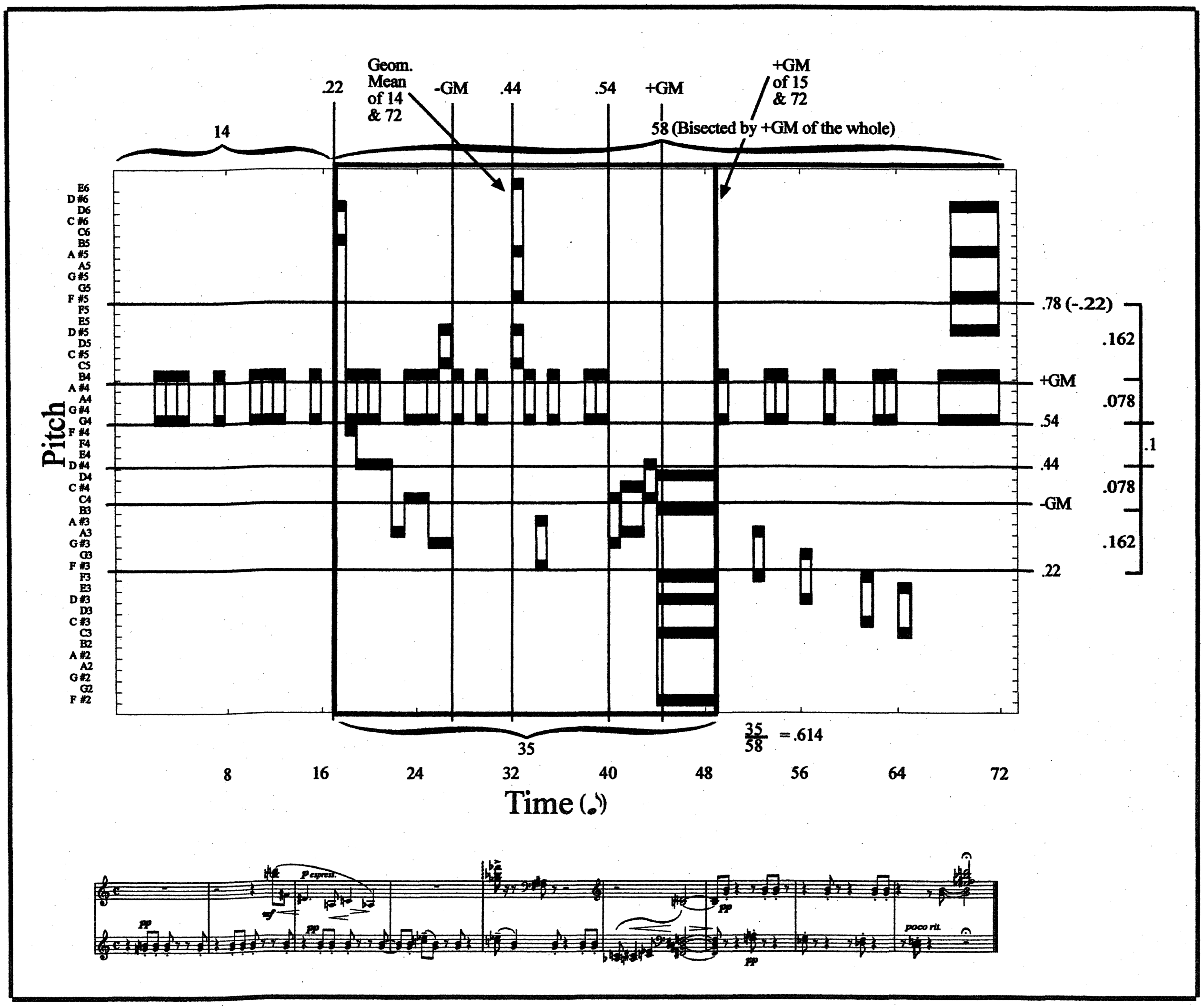

Figure 5: *Pitch-time complex plane*

2.3 Rotation. Note that structurally significant points within each axis, pitch and time, occur at similar relative positions. For instance, the higher of the two ostinato pitches (the opening pitches repeated throughout the piece) occurs at the positive golden section of the pitch (Y) axis while the chord at the nadir of the pitch axis occurs at the positive golden section of the time (X) axis. Notice also the chain of thirds just prior to the above chord commences at .54 of the total time duration of the pitch while the bottom pitch of the above ostinato occurs at .54 of the total pitch span. There exist other such proportional rotations of the pitch axis onto the time axis as illustrated in Figure 6 below.

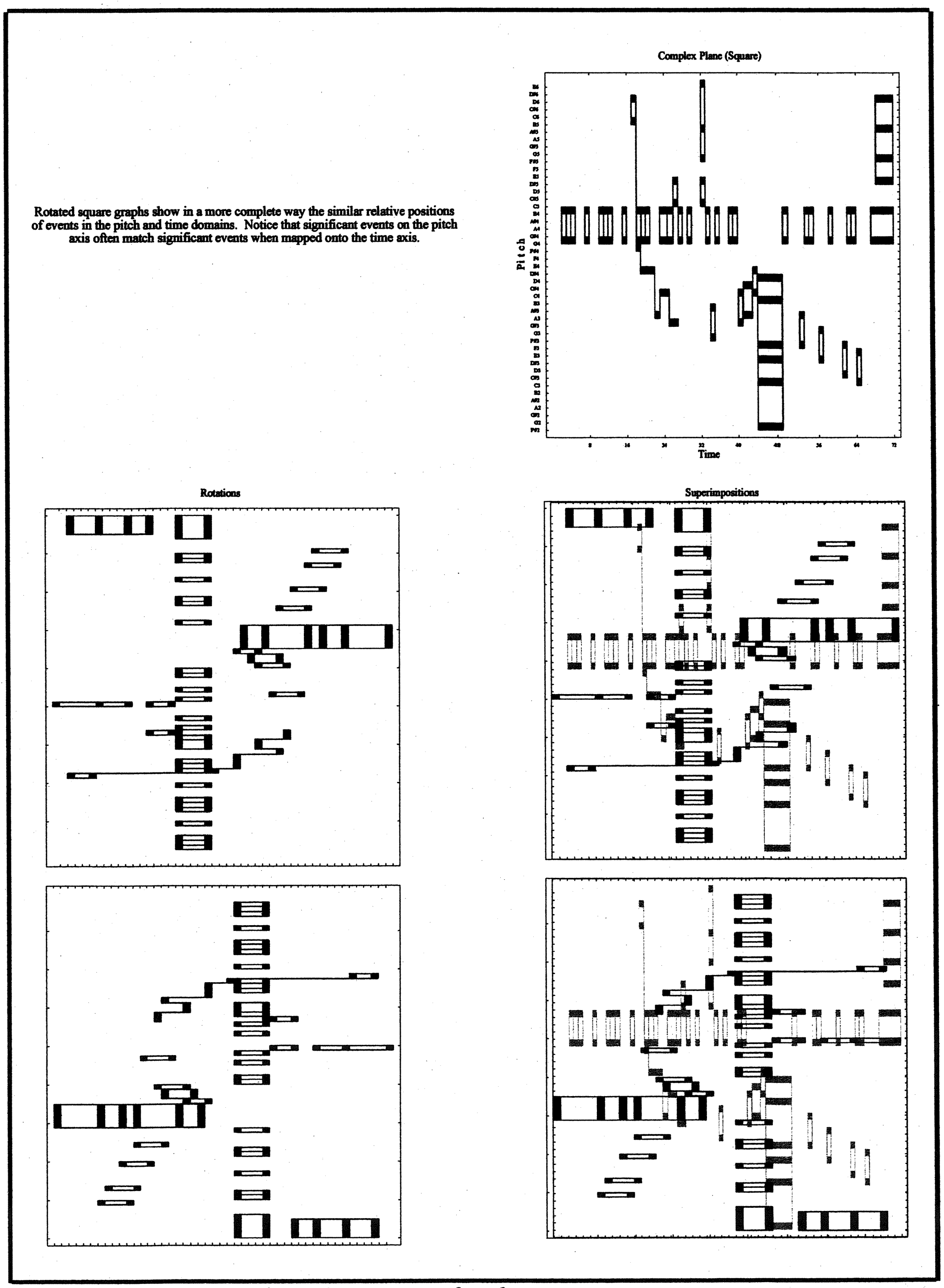

Figure 6: *Rotations of pitch axis onto time axis*

2.4 Symmetry. Schoenberg further emphasizes the significance of the above structural points by focusing bilaterally symmetrical events on them wherein they become the axis of symmetry. For instance, the apex of the piece is an axis of symmetry in the time axis of the beginning of the piece and the final chord in the descending thirds chain near the end of the piece (figure 7). Also, the final chord in the ascending thirds chain just prior to the chord containing the nadir pitch produces the axis of symmetry for the onset of the first pitches outside of the ostinato and the decay (the end) of the final chord. The first pitches outside of the ostinato also find a reflection over the golden section point of duration in the form of the first chord in the ascending third chain. Schoenberg thus relates the first unique event to events leading to the unique chord at the golden section of duration and at the nadir of pitch.

Both self-similar symmetry as well as symmetry similar to the above (centered around significant structural points) is found in the pitch domain. When the pitch content of the music is delineated into sections with the guidance of the time proportions above, symmetry within the pitch domain becomes evident. The pitch content of the first 30 eighth notes of the piece exhibit two such symmetries:

- The local apex and nadir are reflections bisected by the positive golden mean of the total pitch range.
- The local nadir and the highest pitch below the ostinato are bisected by the negative golden mean of the total pitch range.

Note other symmetrical pitch structures centered on the positive and negative golden mean of total pitch range below. Self-symmetrical structures such as that of eighth notes 31-40 are often bounded by structurally important (golden mean) points within the pitch domain as illustrated below.

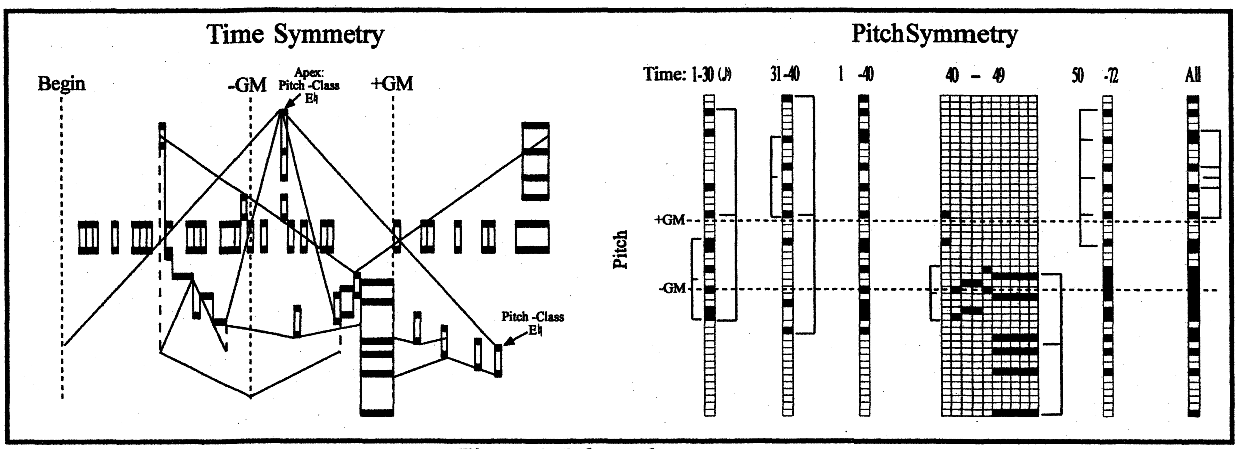

Figure 7: *Bilateral symmetry*

2.5 Pitch Statistics. Schoenberg also distributes pitch-classes to color unique events. The bar charts below illustrate the heavy use of pitch-classes G-natural and B-natural, the ostinato pitches as well as the statistically secondary pitch-classes C, D, D-sharp, F, F-sharp, and A-sharp, which are used within the golden mean chord, the first gesture outside of the ostinato, and the apex chord (but not the apex pitch) thus linking those temporally removed events contextually through pitch material (and extreme registral placement). Notice from Figure 8 below placement of the statistically lowest total duration pitch-classes, C-sharp, E, G-sharp, and A, labeled within the music below. The highest pitches and lowest pitches of the descending and ascending thirds chains are most unique pitches contextually and add weight to the gestures leading into and traveling away from the structurally important chord of the golden mean. Note that Schoenberg places the E-natural (F-flat), the least used pitch-class, as the highest pitch sounded and as a member of the final descending chain of thirds chord, thus singling them out as extremely important events. Notice also (labeled in Figure 8) that each of those pitches is equidistant from the other and the very beginning of the piece, obtaining yet another bilateral symmetry.

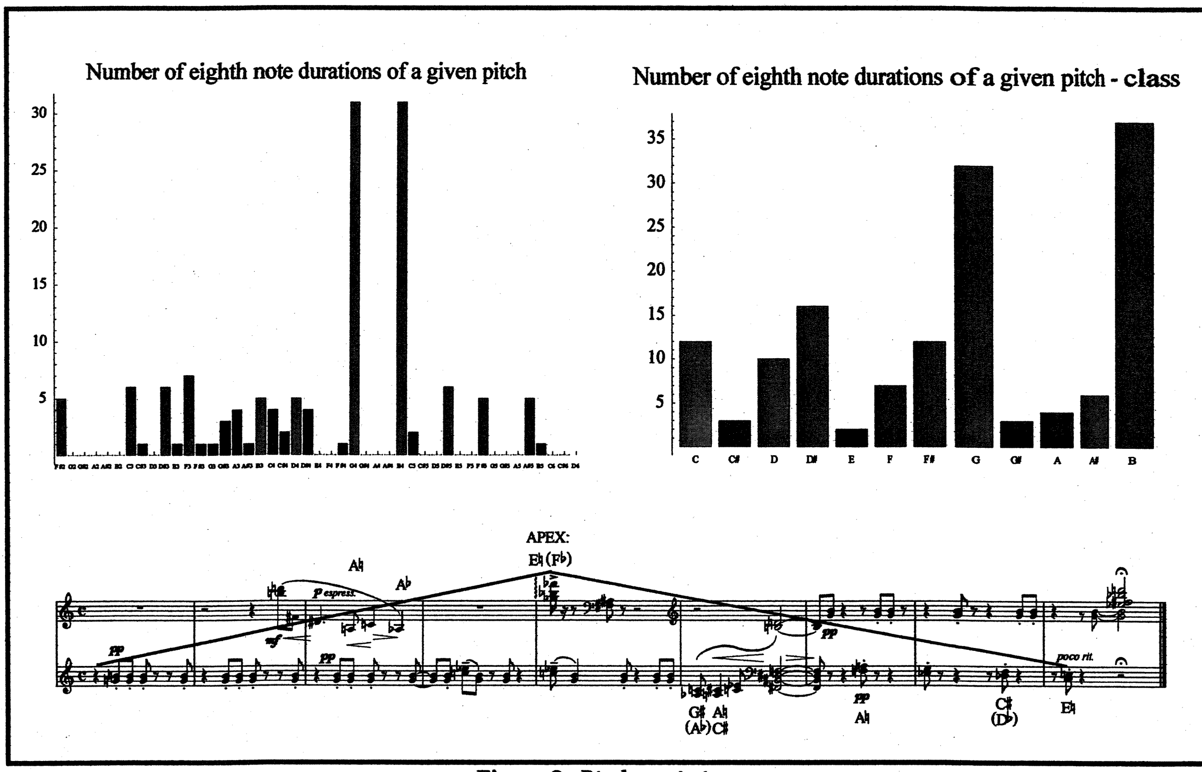

Figure 8: *Pitch statistics*

3. Conclusions and Similarities

We see from the above analysis the architectural genius shared by these two composers of music. Bach and Schoenberg were not merely interested in the surface details of their constructions as often preoccupies the music theorist but upon the self-referencing structure and proportion within and between their multiple dimensions. Bach's construction seems occupied with the internal temporal proportions while Schoenberg's more modern structure exhibits proportions similar both within and between the pitch and temporal dimensions through rotation and symmetry. Note that Bach's construction in the pitch domain unfolds in a very typical (for the time period) linear fashion perhaps not requiring such rigor proportionally to hold the dimension of pitch together as opposed to the carefully proportioned architectural pitch-time structure that Schoenberg requires to tie together his more or less nonlinear unfolding. Bach places alternations of tonic (D minor) and dominant (A major) pitch-class material to exhibit both opposition and unification within and between sections of his structure while Schoenberg's placement of contextually unique pitch material similarly opposes and unifies otherwise temporally nonlinear gestures within his construction.
(The analytical tool of the pitch-time complex plane is found first in the masterful groundbreaking theoretical work *Sonic Design* of Pozzi Escot and Robert Cogan [2].)

References

[1] Hofstadter, Douglas R., *Godel, Escher, Bach: an Eternal Golden Braid (2nd edition)*, New York: Basic Books, 1999.

[2] Cogan, Robert and Escot, Sonic Design: *The nature of Sound and Music,* Cambridge, MA: Publication Contact International, 1984.

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