Nodal Lines and Quasi-Space Filling Curves

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Abstract

Space-filling curves provide a graceful example of how mathematical objects can appeal to one's artistic sense. Similarly, Chladni visualization of a vibrating membrane renders artistically interesting plane curves, or nodal sets, of extreme mathematical value. We explore the nodal lines of particular eigenfunctions of the Dirichlet problem for the Laplace equation in a square domain, and we present a non-recursive family of simple curves which fills the space at the limit. This family of quasi-space filling curves provides a visually interesting application of the nodal set problem, which we also consider in the context of musical vibrations.

1. Introduction

Space-filling curves can be artistically fascinating, for example in computer art, as well as useful in practical applications, such as for grid generations. Another class of plane curves of both artistic and mathematical interest can be created by means of experiments, where the vibrations of a membrane are visualized using sand, which positions itself along the parts of the membrane at rest. Amazingly enough, these patterns correspond to the zero level sets, also known as *nodal lines*, of the eigenfunctions of a Dirichlet problem for the Laplace operator.

Images of these nodal lines are readily available in literature, and can be thought of as visualizations of sound. We are interested in exploring the relationship between space-filling curves and nodal curves. A space-filling nodal curve can be considered to represent the absence of vibration. In particular, we consider the question "Is there a space-filling nodal line?" or the more catching but not rigorous questions *What does the absence of sound look like? What is the image of a vibration that does not vibrate?*

Examples of space filling curves are often constructed as fractals, and can be applied in array-based algorithms and data structures. Many of these examples, are constructed as a limit of a family of curves. Following this idea, we generate a family of curves, which we call *quasi-space filling* as they form a sequence of curves which will fill the space at the limit, although we do not claim they will converge to a space-filling curve.

2. Nodal Sets in Music

Sound can be thought of as a combination of sine waves or pure tones. The more complex tones created by objects vibrating at different frequencies are combinations of these pure tones, expressed mathematically as the eigenfunctions of differential equations, as we shall explore in the next

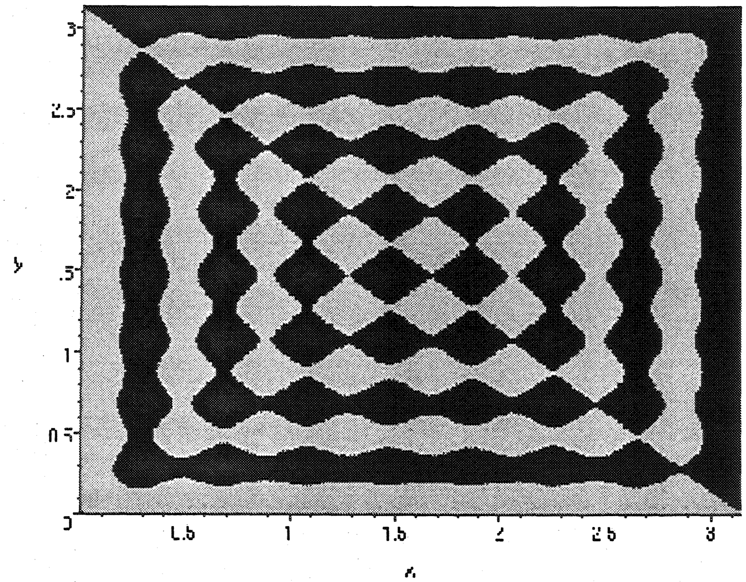

Figure 1: $\delta_j = \frac{1}{2^j+1}, \alpha_j = 2^j, j = 4$

section. Studying the vibrations of specific shapes leads to predictions of the sounds created, giving a mathematical way of visualizing the shapes of sound.

The theory of vibrating strings has been well studied, with early mentions by Galileo and Mersenne (among others) of the musical interpretations of wave frequencies of vibrations. The ancient Greeks knew that the sounds emitted by a vibrating string are dependent upon the nodes or positions at rest. In the 18th century Bernoulli and Euler did considerable work on vibration and elasticity problems, at times in rivalry with d'Alembert, who also studied the wave equation associated to vibrating strings. Bernoulli was able to determine that the musical notes of a stringed instrument are the composition of an infinite number of simple vibrations.

Chladni figures were the first extensive experimentation into the vibrations of two dimensional systems. Mathematical theory of vibrations of surfaces had been studied before Chladni, for example in Euler and Bernoulli's work on elasticity. In addition Chladni's experiments recall those mentioned by Galileo in *Two New Sciences*, regarding patterns made on a brass plate corresponding to the tones emitted.

However German physicist Ernst Chladni's experiments in 1787 on the patterns formed by vibrations were the most comprehensive. These figures drew considerable interest during Chladni's live demonstrations, and impressed Napoleon, who offered a prize for the investigation of a mathematical theory to explain the experimental results. This prize was won by French mathematician Sophie Germain in 1816.

In his experiments, Chladni covered thin metal plates with sand and caused them to vibrate by bowing, which sends the sand to the portions of the plate that are at rest. The resulting patterns represent the nodes of vibration, or nodal lines of the plate. Many of the Chladni patterns give visually pleasing designs which correspond to the zero level sets of eigenfunctions of associated eigenvalue problems (see for example [6]). From the designs, one can picture the shape of the sounds emitted by vibrations of varying frequencies. The symmetry patterns, determined by the shape of the plate and the periodicity of the eigenfunctions, can be quite striking. Chladni experiments have been used to determine ideal shapes for musical instruments, including drums and violins.

In the following sections, we look at eigenfunctions for a square membrane with fixed boundary. The eigenvalue problem presented arises when considering a vibrating homogeneous square membrane, and can be thought as an ideal model for studying the vibration of a square drum.

3. Nodal Sets of Eigenfunctions

We consider the case of a vibrating membrane in the form of the square $[0, \pi] \times [0, \pi]$. We can argue mathematically that in order to find a family of curves that will fill the domain at the limit, we need to look for eigenfunctions corresponding to a sequence of eigenvalues tending to infinity and having simple curves as nodal lines. (We refer to the Appendix below for all the mathematical details, notation and choices used in this section.)

We use the computer algebra systems Scilab and Maple to explore the zero level sets of eigenfunctions. Following [2, page 455], we choose the sequence of eigenvalues $\lambda_j = 1 + \alpha_j^2$ with $\alpha_j = 2^j$ and as corresponding eigenfunctions $u_j(x, y) = \sin x \, \sin(\alpha_j \, y) + (1 - \delta_j) \, \sin(\alpha_j \, x) \, \sin y$, with $\delta_j \to 0$. In this case, it is easy to see that the eigenfunctions will have a single nodal curve (see Figures 7 and 8 in [2] page 456).

We experimented with several different expressions for the sequence δ_j, and we found that to obtain simple curves we need a relationship between the decay to 0 of δ_j and the rate at which 2^j tends to infinity (as we argue theoretically at end of the Appendix). In particular, if δ_j tends to 0 not rapidly enough the nodal lines are not simple curves even for very small values of j (obviously, one might expect a high rate of failure for j large since the singularities β_{il}^j could conceivably accumulate around 1). On the other hand, we have to balance our need of a rapid decay with the search for interesting and "visible" simple curves. Our experimentations lead us to choose sequences inversely proportional to α_j.

Figure 1 presents the simple curve corresponding to the choice $\delta_j = \frac{1}{2^j+1}$, for $j = 4$, where the shading enhances the two sets $u_4(x, y) > 0, u_4(x, y) < 0$, and emphasizes that these are simply connected domains.

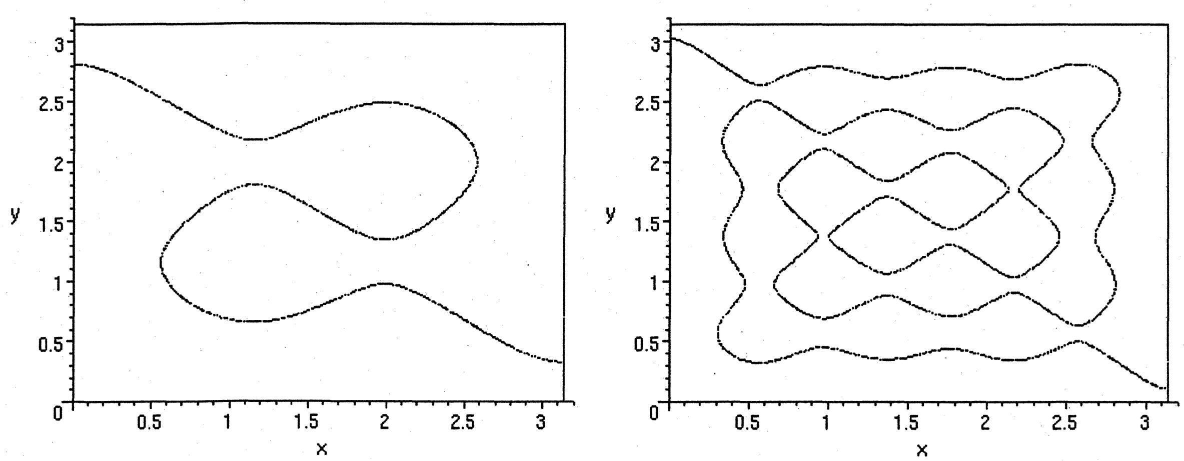

Figure 2: $\delta_j = \frac{1}{2^j}, \alpha_j = 2^j, j = 2, j = 3$

In Figures 2, 3, and 4 we show the first few members of the family of simple curves that we created by picking $\delta_j = \frac{1}{2^j}$. The pictures give a good idea of the behavior of the curve (that is of the "silent" part of the membrane) as j increases. A 180 degrees rotational symmetry is also evident in our curves.

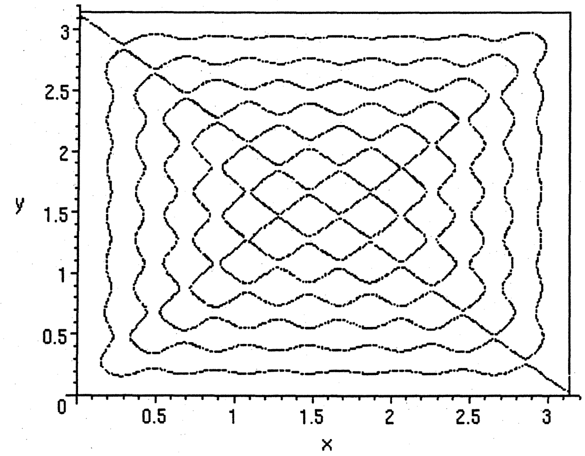

Figure 3: $\delta_j = \frac{1}{2^j}, \alpha_j = 2^j, j = 4$

The theory also helps us to quantify the rate at which our curves fill the square. In fact, from [5] we know that the length of a nodal line is proportional to $\lambda^{\frac{1}{2}}$, so in our case the length of our curves tends to ∞ at the rate of $2^{\frac{j}{2}}$.

A The Mathematics of the Family

We consider the following two-parameter family of functions

$$u_j(x,y) = \sin x \, \sin(\alpha_j \, y) + \epsilon_j \, \sin(\alpha_j \, x) \, \sin y \quad \text{for } (x,y) \in [0,\pi] \times [0,\pi]$$

which are solutions to the Dirichlet eigenvalue problems for the Laplace equation in two dimensions:

$$\begin{cases} u_{xx} + u_{yy} = -\lambda_j u, & \text{in } [0,\pi] \times [0,\pi] \\ u = 0, & \text{on } \{0,1\} \times [0,\pi], \text{ and } [0,\pi] \times \{0,1\} \end{cases}$$

with $\lambda_j = 1 + \alpha_j^2$. It is well-known [2, pages 300-302] that if α_j is an integer then $u_j(x,y)$ is an eigenfunction for a square membrane. The level set $u_j(x,y) = 0$ locates the points at rest on the membrane when it is vibrating with frequency λ_j.

Quite a bit is know regarding the eigenvalues and eigenfunctions for the Dirichlet eigenvalue problem which we can use as guidelines for our construction. In particular, if the nodal line $u_j(x,y) = 0$ is a simple curve it will divide our square into two simply connected domains, say Ω_j^1, Ω_j^2. We then have that the restriction of u_j to Ω_j^1 (respectively to Ω_j^2) is an eigenfunction for Ω_j^1 (Ω_j^2) which does not change sign. Using the fact that only the first eigenfunction does not change sign ([2, page 451]), we can conclude that λ_j is the first eigenvalue for the Dirichlet eigenvalue problem in Ω_j^1 (respectively in Ω_j^2). But, the first eigenvalue for a simply connected domain is comparable to the radius of the largest ball inscribed inside the domain [3, Theorem 1.5.10 page 32]; since (again from [2, page 451]) we know that the largest ball inscribed in Ω_j^1 (respectively, Ω_j^2) has radius $r \leq \frac{k_{0,1}^2}{\lambda_j}$ (here $k_{0,1}$ is the first zero of the zero-th Bessel function), if $\lambda_j \to \infty$ we have a sequence of curves that fill the space at the limit.

The Implicit Function Theorem tells us that if $\nabla u_j(x,y) \neq 0$ in $(0,\pi) \times (0,\pi)$ then the nodal line $u_j = 0$ describes a simple curve. Therefore, our search translates to finding α_j, ϵ_j such that the three following conditions are NOT satisfied simultaneously:

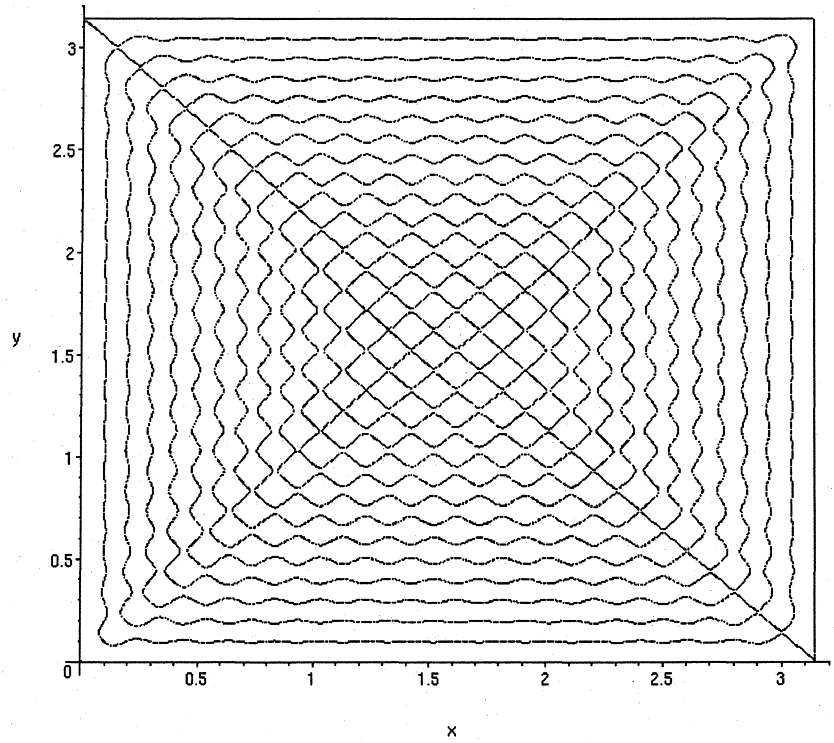

Figure 4: $\delta_j = \frac{1}{2^j}, \alpha_j = 2^j, j = 5$

$$\cos x \sin(\alpha_j\, y) + \epsilon_j \alpha_j \sin y \cos(\alpha_j\, x) = 0 \tag{1}$$
$$\alpha_j \sin x \cos(\alpha_j\, y) + \epsilon_j \cos y \sin(\alpha_j\, x) = 0 \tag{2}$$
$$\sin x\, \sin(\alpha_j\, y) + \epsilon_j\, \sin(\alpha_j\, x)\, \sin y = 0. \tag{3}$$

One can easily check the following claims:

Claim 1 *If* $(x,y) \in (0,\pi) \times (0,\pi)$ *and* $u_j(x,y) = 0$, *then* $\sin(\alpha_j\, x) = 0$ *if and only if* $\sin(\alpha_j\, y) = 0$;

Claim 2 *If* $(x,y) \in (0,\pi) \times (0,\pi)$ *and* $u_j(x,y) = 0$, *then* $\nabla u_j \neq 0$ *if* $\sin(\alpha_j\, x) = 0, \sin(\alpha_j\, y) = 0$ *(note: in this case one has* $|\nabla u_j|^2 = \epsilon_j^2 \alpha_j^2 \sin^2 y + \alpha \sin^2 x \neq 0$*);*

Claim 3 *If* $(x,y) \in (0,\pi) \times (0,\pi)$ *and* $u_j(x,y) = 0$, *then* $\nabla u_j \neq 0$ *if one of the following conditions holds:* $\cos(\alpha_j y) = 0, \cos y \neq 0$ *or* $\cos(\alpha_j y) \neq 0, \cos y = 0$ *or* $\cos(\alpha_j x) = 0, \cos x \neq 0$ *or* $\cos(\alpha_j x) \neq 0, \cos x = 0$*;*

Claim 4 *If* $(x,y) \in (0,\pi) \times (0,\pi)$ *and* $u_j(x,y) = 0$, *then* α_j *an even integer and* $\nabla u_j = 0$ *imply that* $\cos(\alpha_j y) = \cos y = 0$, *(or* $\cos(\alpha_j x) = 0, \cos x = 0$*) cannot hold;*

Claim 5 *If* $(x,y) \in (0,\pi) \times (0,\pi)$ *and* $\epsilon_j^2 \neq 1$, *then* $u_j(x,y) = 0$, $\nabla u_j = 0$, *and* $\cos(\alpha_j y) = \cos y = \cos(\alpha_j x) = \cos x = 0$ *cannot hold.*

As a consequence of the above claims, we have that as long as $\epsilon_j^2 \neq 1$ and α_j is an even integer, we can just consider the case where the sine and cosine functions at $x, \alpha_j x$ and at $y, \alpha_j y$ are all non-zero. We then combine equations (1) and (2) with equation (3) to obtain the equivalent set of conditions:

$$\alpha_j \tan x = \tan(\alpha_j\, x) \tag{4}$$

$$\alpha_j \tan y = \tan(\alpha_j\, y) \tag{5}$$

$$\sin x\, \sin(\alpha_j\, y) + \epsilon_j\, \sin(\alpha_j\, x)\, \sin y = 0. \tag{6}$$

Clearly, since tangent is an increasing function where defined, if α_j is an even integer, $\alpha_j \tan t = \tan(\alpha_j\, t)$ can hold at most at α_j points (let's say $z_i^{\alpha_j}$ for $i = 1, \dots, \alpha_j$), so there are at the most α_j^2 values (say $-\beta_{ik}^j \equiv \dfrac{\sin z_i^{\alpha_j}\, \sin(\alpha_j\, z_k^{\alpha_j})}{\sin(\alpha_j\, z_i^{\alpha_j})\, \sin z_j^{\alpha_j}}$ for $i, k = 1, \dots, \alpha_j$), for which equations (4), (5) and (6) all hold true, so as long as we choose $\epsilon_j \neq \beta_{ik}^j$ our level curves will be simple closed curves. Moreover, if we restrict ourself to the case $\epsilon_j > 0$ there are even fewer possible "unfavorable" choices.

We finally remark that if we consider any of the $z_i^{\alpha_j} \in (0, \pi)$, by the proprieties of the tangent function, and due to the fact that α_j is an even integer, one of the remaining $z_l^{\alpha_j}$, for some l, must be equal to $\pi - z_i^{\alpha_j}$ (since α_j an even integer implies $\alpha_j \tan(\pi - z_i^{\alpha_j}) = \tan(\alpha_j(\pi - z_i^{\alpha_j}))$). Therefore, we have $\beta_{il}^j = 1$, which tells us that if we keep ϵ_j close to 1 (for example by taking $\epsilon_j = 1 - \delta_j$ with δ_j decaying to 0 "fast enough") we will avoid the "unfavorable" choices.

The picture to the right in Figure 5 shows the simple curve given by the nodal set for the choice $\epsilon_j = 1 - \frac{1}{2^j}, \alpha_j = 2^j, j = 4$; in the one to the left we shade the domains $u_j > 0$, $u_j < 0$ differently, to highlight that they are in fact simply connected.

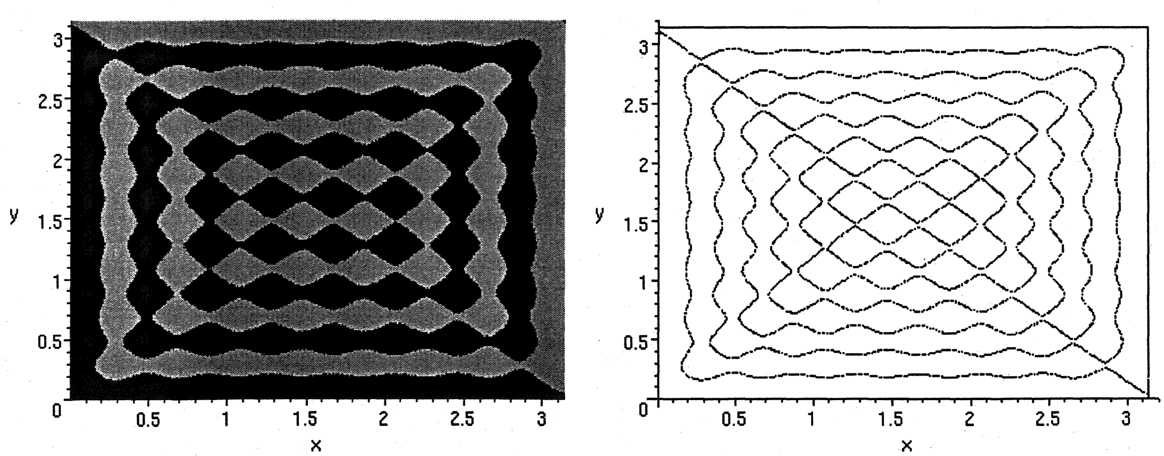

Figure 5: $\epsilon_j = 1 - \frac{1}{2^j}, \alpha_j = 2^j, j = 4$

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