Fractal Tilings Based on Dissections of Polyominoes

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Abstract

Polyominoes, shapes made up of squares connected edge-to-edge, provide a rich source of prototiles for edge-to-edge fractal tilings. We give examples of fractal tilings with 2-fold and 4-fold rotational symmetry based on prototiles derived by dissecting polyominoes with 2-fold and 4-fold rotational symmetry, respectively. A systematic analysis is made of candidate prototiles based on lower-order polyominoes. In some of these fractal tilings, polyomino-shaped holes occur repeatedly with each new generation. We also give an example of a fractal knot created by marking such tiles with Celtic-knot-like graphics.

1. Introduction

Fractals and tilings can be combined to form a variety of esthetically appealing constructs that possess fractal character and at the same time obey many of the properties of tilings. Previously, we described families of fractal tilings based on kite- and dart-shaped quadrilateral prototiles [1], v-shaped prototiles [2], prototiles that are segments of regular polygons [3], and prototiles derived by dissecting polyhexes [4]. Many of these constructs may be viewed online [5]. These papers appear to be the first attempts at a systematic treatment of this topic, though isolated examples were earlier demonstrated by M.C. Escher [6] and Peter Raedschelders [7].

In Grünbaum and Shephard's book *Tilings and Patterns* [8], a tiling is defined as a countable family of closed sets (tiles) that cover the plane without gaps or overlaps. The constructs described in this paper do not for the most part cover the entire Euclidean plane; however, they do obey the restrictions on gaps and overlaps. To avoid confusion with the standard definition of a tiling, these constructs will be referred to as "*f*-tilings", for fractal tilings.

The tiles used here are "well behaved" by the criterion of Grünbaum and Sheppard; namely, each tile is a (closed) topological disk. Most of the *f*-tilings explored in References 1-5 are edge-to-edge; i.e., the corners and edges of the tiles coincide with the vertices and edges of the tilings. However, they are not "well behaved" by the criteria of normal tilings; namely, they contain singular points, defined as follows. Every circular disk, however small, centered at a singular point meets an infinite number of tiles. Since any *f*-tiling of the general sort described here will contain singular points, we will not consider singular points as a property that prevents an *f*-tiling from being described as "well behaved". These *f*-tilings provide a rich source of unique fractal images and also possess considerable recreational mathematics content.

The prototiles considered in this paper are derived by dissecting polyominoes. A polyomino is a shape made by connecting squares in edge-to-edge fashion. Following common usage, the first nine polyominoes will be called monomino, domino, tromino, tetromino, pentomino, hexomino, heptomino, octomino, and nonomino, respectively. Polyominoes made from a number of squares $n > 9$ will be called n-ominoes. For a discussion of different types of polyominoes and conventional tilings using them, see References 8 and 9. Most polyominoes have adjacent straight-line segments longer than the edges of the constituent squares, as shown in Figure 1. For this reason, there are relatively few true edge-to-edge *f*-tilings using prototiles created by dissecting polyominoes. In order to provide a richer variety of examples, pseudo-edge-to-edge *f*-tilings will be considered as well. In these cases, the edges of the constituent squares are considered to define pseudo-edges of the polyominoes, as shown in Figure 1.

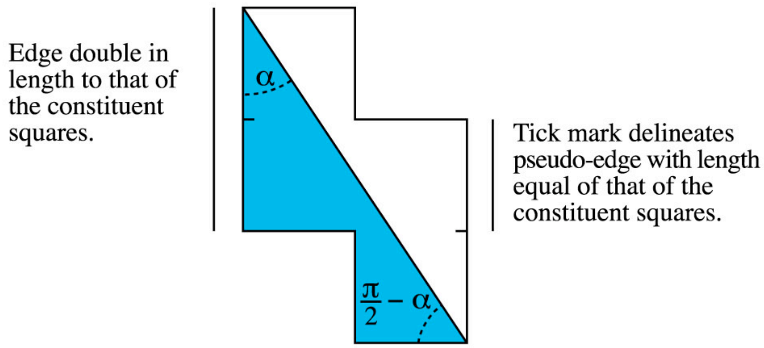


Figure 1: *A tetromino with tick marks indicating pseudo-edges and irrational angle α.*

Each prototile has one or two long edges and two or more short edges. The angles between the long edges and adjacent short edges are for the most part irrational, but sum to a multiple of $\pi/2$, as shown in Figure 1. The *f*-tilings are constructed by first matching the long edges of identical prototiles, then cloning the prototile, reducing its size by the ratio of the length of a short edge to that of a long edge, fitting multiple copies of these smaller tiles around the first generation of tiles according to a matching rule, and finally iterating this process an infinite number of times. For a given *f*-tiling, every tile is similar to the single prototile. The group of first-generation tiles forms the dissected polyomino. In practice, the appearance of the overall tiling changes little after 4-7 generations, since the tiles become extremely small relative to the first-generation tiles. The figures shown here are constructed by iterating until the individual tiles become very small on the scale of the page size. Fractal tilings may also be constructed that possess more than one prototile, but these will not be considered here.

2. Candidate Prototiles

Three requirements simplify the search for polyomino-based prototiles that allow *f*–tilings.

1. The generating polyomino must have 2-fold or 4-fold rotational symmetry. While polyominoes without rotational symmetry may be used, they generate no new prototiles, and therefore they do not need to be considered. While not proven here, this is readily apparent by examining a table showing all polyominoes possible for a given number of squares. In addition, mirrored variants of polyominoes are not considered to generate distinct prototiles, as they would result in *f*-tilings that are an overall mirror of the *f*-tilings constructed from non-mirrored variants.

2. For a prototile generated by bisecting a polyomino, each bisecting line, which will form the long edge of the prototile, must originate and terminate at corners of the polyomino and pass through the centroid of the polyomino. If the endpoints aren't at corners or pseudo-corners, the short edges will not all be of the same length. The long edges of the prototile must be longer than the short edges or pseudo-edges of the prototile.

3. For a prototile generated by dissecting a polyomino into four equal parts, each dissecting line, which will form one long edge of the prototile, must run from the centroid of the polyomino to a corner. The four dissecting lines are related to each other by rotations of 90° about the centroid. Again, the long edges of the prototile must be longer than the short edges or pseudo-edges of the prototile. For reasons shown below, the number of short edges or pseudo-edges must be even. This rules out any polyomino made up of $4n + 1$ squares [10]. The only other polyominoes with 4-fold symmetry are made up of $4n$ squares, so only these need be considered.

Figure 2 shows candidate polyominoes made up of 1 to 5 squares and prototiles that meet these criteria. Prototiles colored green and red allow *f*-tilings, while those colored black do not. Only green prototiles allow true edge-to-edge *f*-tilings.

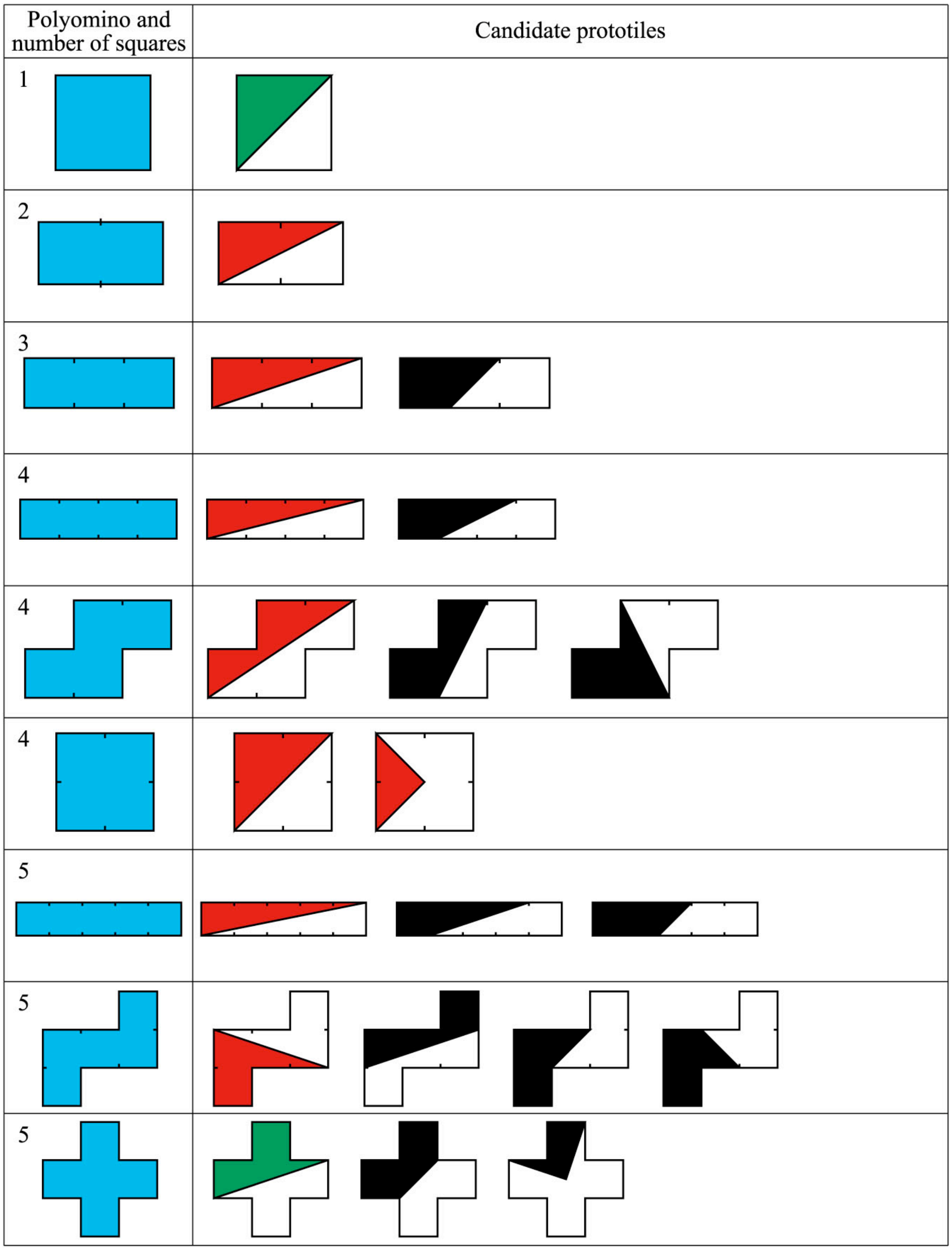


Figure 2: *Candidate prototiles for polyominoes made up of 1 to 5 squares. Green ones tile edge-to-edge and red ones pseudo-edge-to-edge, while the black ones do not tile.*

3. *f*–tilings Based on Dissected Polyominoes

3.1. Bounded *f*-tilings. In this section, we give a number of examples of *f*–tilings constructed from prototiles derived by dissecting polyominoes. Our first examples have overall 2-fold rotational symmetry. The starting point for an *f*-tiling of this sort is a pair of tiles that form the generating polyomino. These are surrounded by smaller tiles in (pseudo-) edge-to-edge fashion, with the construction process proceeding iteratively. The only option is whether or not the tiles are mirrored between successive generations, but no mirrored variants are considered here due to space limitations. The process by which *f*-tilings are constructed is described in greater detail in Reference 1.

In Figure 3, we show two examples of *f*-tilings of this sort. The left *f*-tiling is edge-to-edge, with a reduction factor of $1/\sqrt{10}$ between tiles of successive generations. This number is easily obtained by noting that the diagonal is that of three squares in a row and then applying the Pythagorean theorem. (If the squares have edges of length 1, the long edge of the prototile has length $\sqrt{(1^2 + 3^2)}$.) Between successive generations the tiles are rotated by arctan(1/3) ≈ 18.43° (plus multiples of $\pi/2$ as required for fitting a particular edge). The right *f*-tiling in Figure 3 is pseudo-edge-to-edge, with a reduction factor of $1/\sqrt{13}$ (which can be obtained in similar fashion).

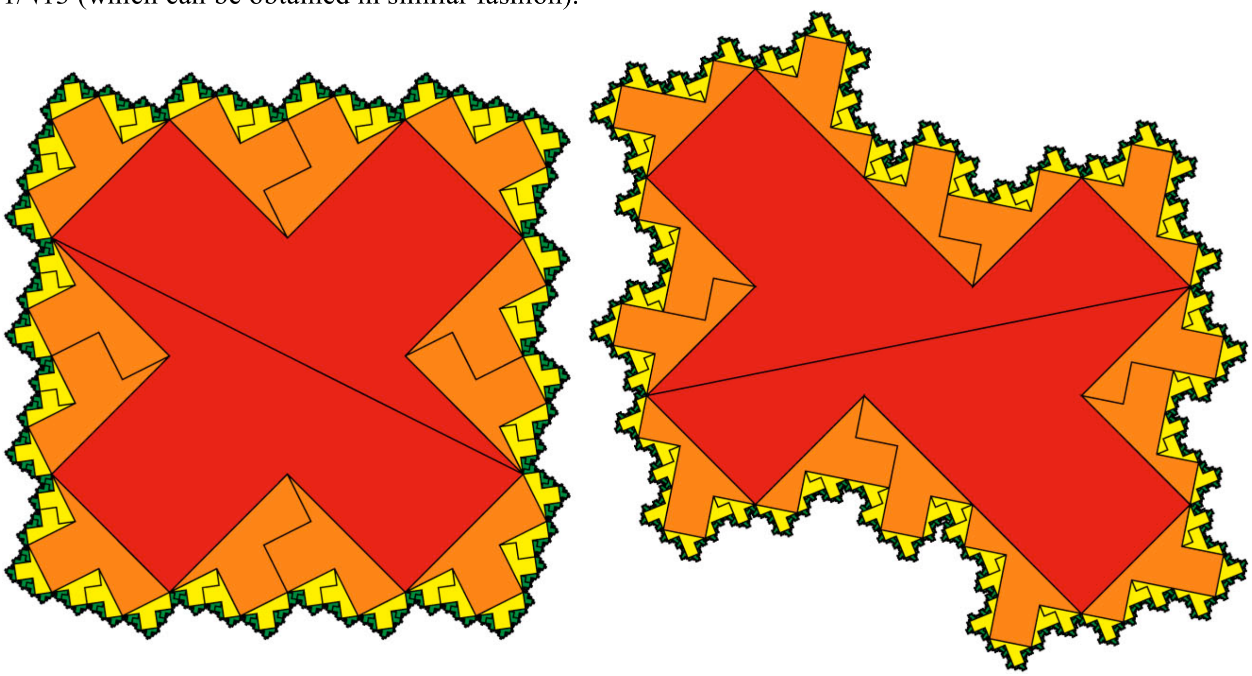


Figure 3: *(a) A true edge-to-edge f–tiling generated from a prototile based on the "X" pentomino.* (b) *A pseudo-edge-to-edge f–tiling generated from a prototile based on a hexomino.*

In general, the boundaries of *f*-tilings are fractal curves, though there are cases in which the boundaries are non-fractal polygons. The boundaries are similar to Koch islands and related constructs, in which a line segment is distorted into multiple smaller line segments, which are in turn distorted according to the same rule, etc. [11].

In Figure 4, we show a 2-fold rotational *f*-tiling in which the boundary is an octagon with two different edge lengths. In order to make the figure more interesting visually, the tiles, which are isosceles right triangles, are decorated with Celtic-knot-like markings. The markings on this *f*-tiling thus form a fractal knots (or fractal links, to use knot terminology more properly) design.

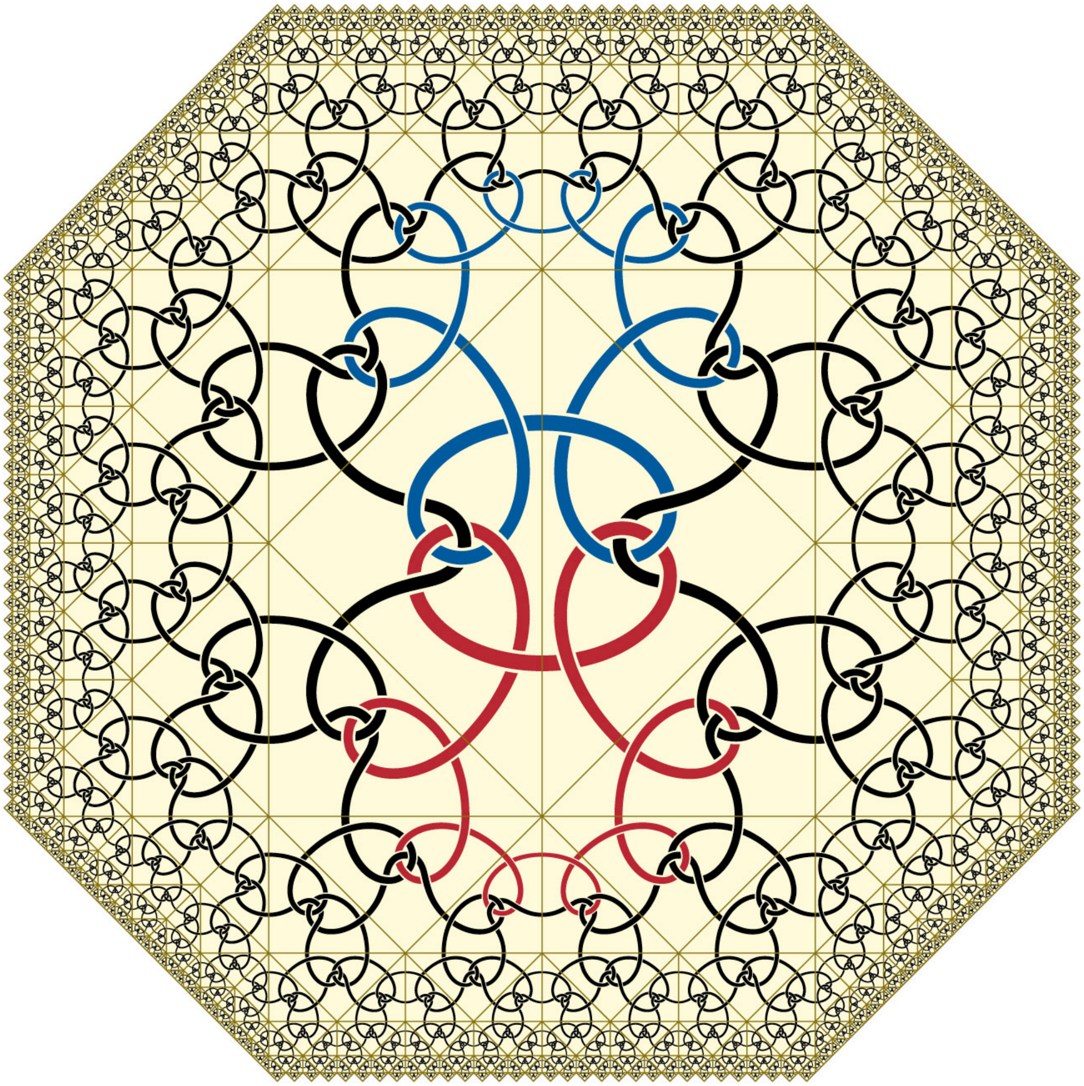


Figure 4: *A true edge-to-edge f–tiling generated from the monomino. In order to make a more esthetically-pleasing figure, the tiles are decorated with markings that create a fractal knots design.*

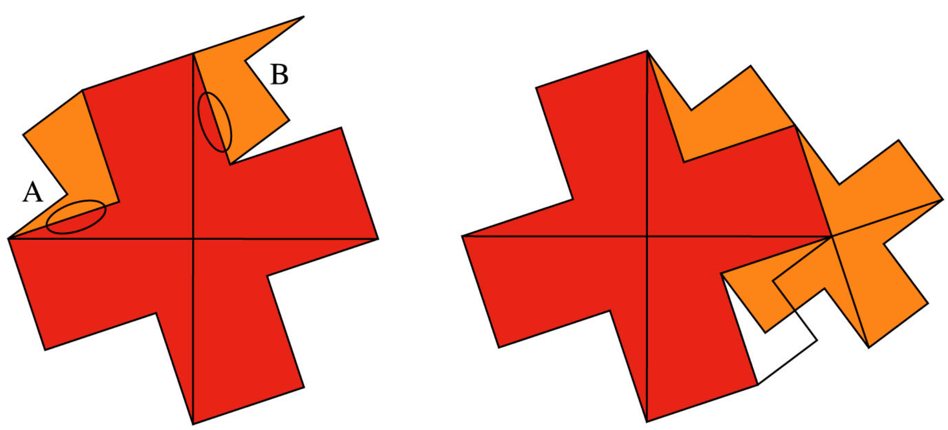
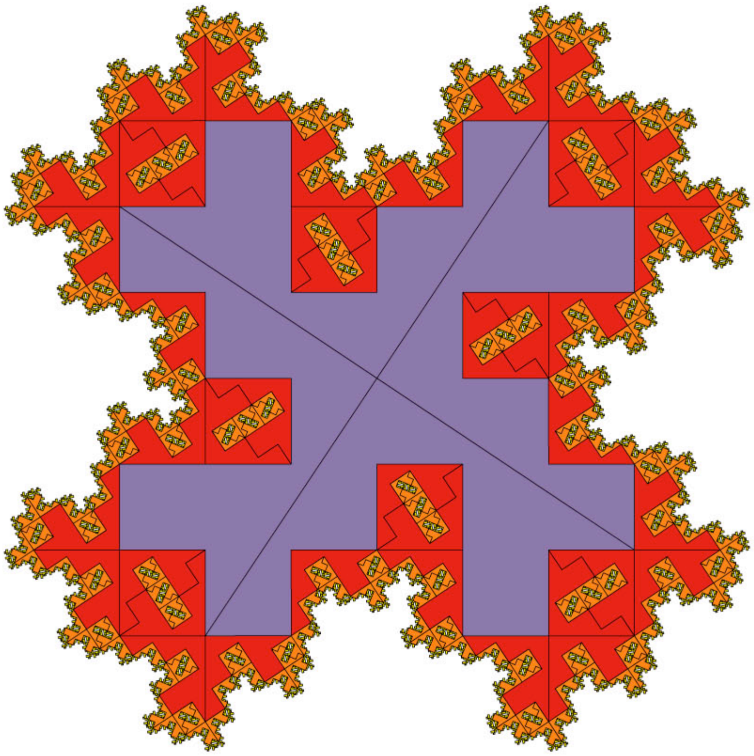


Figure 5: *For prototiles generated from 4-fold polyominoes, there are two choices, A and B, for arranging smaller tiles around large tiles. In order to avoid overlaps using a consistent matching rule, 4-fold prototiles must have an zeven number of short edges.*



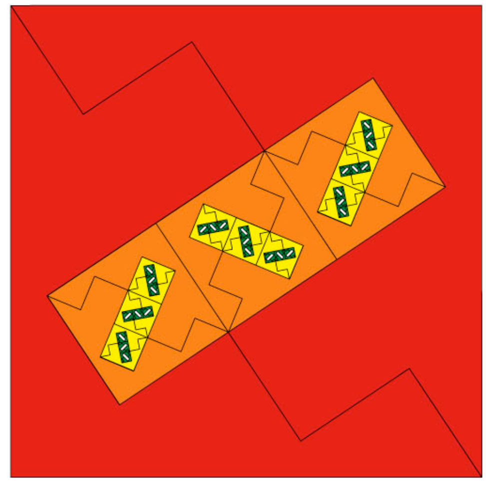
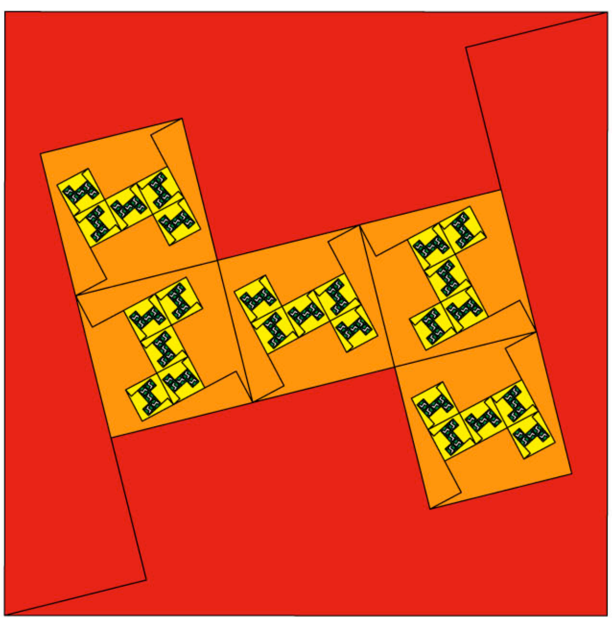


Figure 6: *An f–tiling generated from a prototile based on a 20-omino. Note the formation of "I" tromino-shaped holes. The zoomed-in portion of the f-tiling at right highlights this feature. The scaling and rotation between successive generations of tiles is 1/√13 and ≈ 33.7°.*



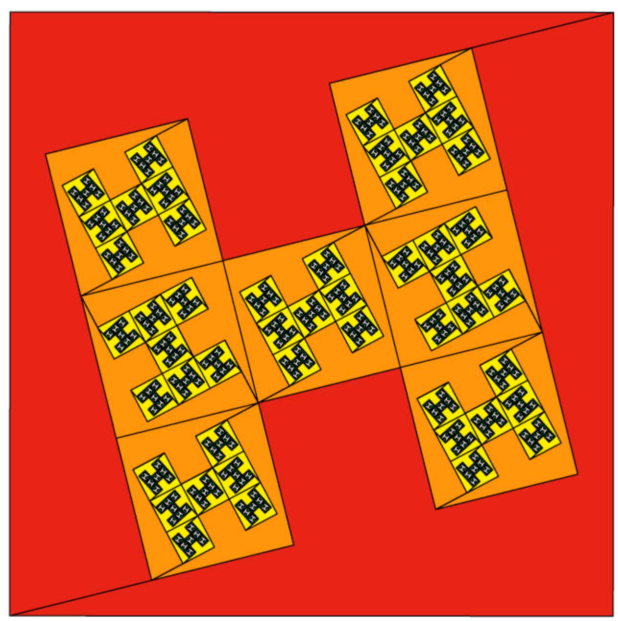


Figure 7: *Portions of two f–tilings generated from prototiles based on 24- and 20-ominoes, respectively. Note the repeated formation of "Z" pentomino- and "H" heptomino-shaped holes. The scaling between successive generations of tiles is 1/√17 for both f-tilings.*

Next we show some examples with overall 4-fold rotational symmetry. For these prototiles, there are two choices of matching rules for arranging tiles of a given generation around tiles of the next larger generation. This is illustrated in the left side of Figure 5, where the two choices are labeled A and B. The right side of Figure 5 shows why the number of short edges cannot be odd if a single matching rule is used. The fact that there are two long edges for each smaller tile requires an even number of short edges for each larger tile if overlaps are to be avoided.

An *f*-tiling of this sort is shown in Figure 6, where the prototile is derived from a 20-omino. The left side of Figure 6 shows the full *f*-tiling, while the right shows a magnification of a hole formed by two tiles fit together within a square region. The hole is shaped like the "I" tromino, and further iterations create three smaller holes of the same sort within each larger hole. This tripling of holes with the same shape will clearly continue *ad infinitum* as additional generations of tiles are added, with the hole just being filled in the infinite limit. Figure 7 shows similar square regions for two additional 4-fold *f*-tilings. The left figure shows "Z" pentomino-shaped holes using a prototile derived from a 24-omino, while the right figure shows "H" heptomino-shaped holes using a prototile derived from a 20-omino.

In some cases, an entire *f*-tiling constitutes a supertile that tiles the plane; the *f*-tilings in Figure 8 possesses this property. Plane filling may also be achieved by using square building blocks such as those shown in Figure 7.

4. Conclusions

We have presented several examples of fractal tilings (*f*-tilings) based on prototiles derived by dissecting polyominoes with 2-fold and 4-fold rotational symmetry. There are an infinite number of polyominoes and an infinite number of *f*-tilings of this sort. However, in general they become increasingly less interesting for higher-order polyominoes due to the fact that the scaling factor between successive generations becomes more extreme. Decoration of tiles with markings to create fractal knots is one means of increasing the visual interest of these constructs. Fractal tilings that result in recurring polyomino-shaped holes have also been demonstrated.

References

[1] Robert W. Fathauer, *Fractal tilings based on kite- and dart-shaped prototiles*, Computers & Graphics, Vol. 25, pp. 323-331, 2001.

[2] Robert W. Fathauer, *Fractal tilings based on v-shaped prototiles*, Computers & Graphics, Vol. 26, pp. 635-643, 2002.

[3] Robert W. Fathauer, *Self-similar Tilings Based on Prototiles Constructed from Segments of Regular Polygons*, in Proceedings of the 2000 Bridges Conference, edited by Reza Sarhangi, pp. 285-292, 2000.

[4] Robert W. Fathauer, *Fractal Tilings Based on Dissections of Polyhexes*, in Renaissance Banff, Mathematics, Music, Art, Culture Conference Proceedings, 2005, edited by Reza Sarhangi and Robert V. Moody, pp. 427-434, 2005.

[5] Robert W. Fathauer, http://members.cox.net/fractalenc/encyclopedia.html.

[6] Bruno Ernst, *The Magic Mirror of M.C. Escher*, Ballantine Books, New York, 1976.

[7] Peter Raedschelders, "Tilings and Other Unusual Escher-Related Prints," in M.C. Escher's Legacy, edited by Doris Schattschneider and Michele Emmer, Springer-Verlag, Berlin, 2003.

[8] Branko Grünbaum and G.C. Shephard, *Tilings and Patterns*, W.H. Freeman, New York, 1987.

[9] Solomon W. Golomb, *Polyominoes*, Princeton University Press, Princeton, New Jersey, 1994.

[10] The only pentomino with 4-fold rotational symmetry yields prototiles with 3 short edges. Adding 4 squares to this pentomino yields prototiles with either 3 or 5 short pseudo-edges. It can easily be seen that adding four squares to any 4-fold polyomino will either add 2 short pseudo-edges, leave the number of short pseudo-edges unchanged, or subtract 2 short pseudo-edges. The number of short pseudo-edges for prototiles is therefore always odd.
[11] H.-O. Peitgen, H. Jürgens, and D. Saupe, *Fractals for the Classroom – Part One*, Springer-Verlag, New York, 1992.

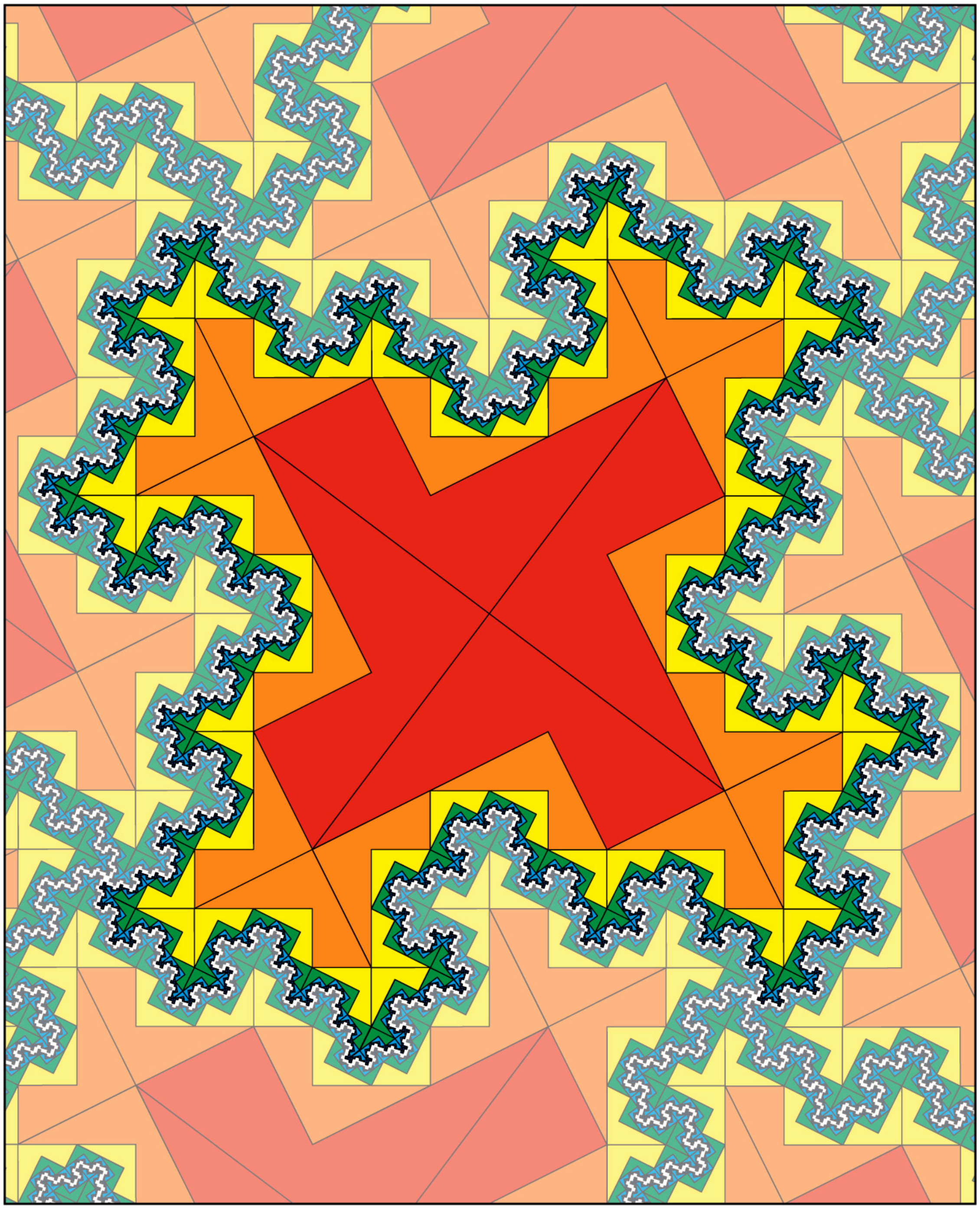


Figure 8: *A plane-filling f–tiling generated from a prototile based on an octomino. The f-tiling was carried through six generations, with each generation a different color.*