The Gemini Family of Triangles

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Abstract

There are a series of triangles in the pentagon/pentagram figure that can be used advantageously in quilting. We are going to investigate these triangles both mathematically and artistically.

Introduction

A regular pentagon *P* with its inscribed "star" (the pentagram constructed from the five diagonals of *P*) Figure 1 can be advantageously used to construct quilt designs that are aesthetically pleasing and mathematically fascinating. Unfortunately, the quilt program's [2] only five point star block was not properly aligned. In this paper we will discuss a correct mathematical construction of *P* and the related quilt designs.

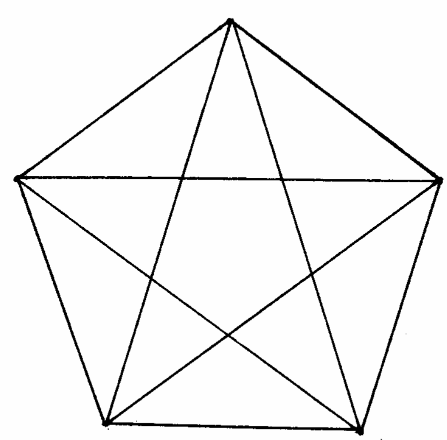

Figure 1: Regular Pentagon with Inscribed Pentagram

Mathematics

The first thing one should notice in figure 1 is that the star subdivides the area bounded by *P* into a small central regular pentagon and ten isosceles triangles circumscribing it. These ten triangles are of two distinct types and indeed all of the thirty-five triangles that can be formed from the sides and diagonals of *P* are of one or the other of these two types.

L. Gordon Plummer says “These triangles were held by the Greeks to be so important that they were named after the two stars in the sky associated with the constellation Gemini, the twins: Castor and Pollux.” [3]

A **Castor** Triangle is an isosceles triangle whose apex angle measures $\pi/5$ radians and consequently has base angles each measuring $2\pi/5$ radians.

A **Pollux** Triangle is an isosceles triangle whose apex angle measures $3\pi/5$ radians and consequently has base angles each measuring $\pi/5$ radians.

The construction of a Castor triangle is the key to inscribing a regular *decagon* (and hence also a regular *pentagon*) in a circle. For the central angle subtended by a side of a regular decagon, is $2\pi/10 = \pi/5$ and consequently the triangle formed by this side and the two radii terminating at the ends of this side is a Castor triangle. As a further consequence, the triangle formed by two diagonals of a regular *pentagon* emanating from the same vertex and the side of the regular *pentagon* that terminates these diagonals is also a Castor triangle [1].

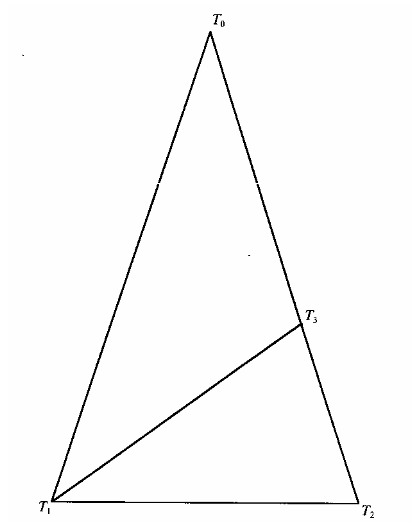

Figure 2: Mother Castor and Castor and Pollux twins

Let $[T_1,T_3]$ be the bisector of the base angle of $C_0 = [T0,T1,T2]$ with vertex T_1 and foot T_3 on $[T_2,T_0]$. This bisector subdivides C_0 into two isosceles triangles, $C_1 = [T_1, T_2, T_3]$ which is similar to C_0 and $P_1 = [T_3, T_0, T_1]$. The apex angle of P_1 measures $3\pi / 5$ radians and hence is a Pollux triangle. The common side $[T_1,T_3]$ of both triangles is equal to $[T_1,T_2]$ (which is the side of the decagon) and $[T_3,T_0]$. This configuration, consisting of two isosceles triangles, one a Castor and the other a Pollux, which share a common side with the apex angle of each contiguous with a base angle of the other, we call a pair of **Gemini Twins** whose **mother** is the Castor triangle from which they were derived and to which they return when their common side (the bisector) is erased.

Since C_0 and C_1 are similar and isosceles and P_1 is isosceles we can write
$\varphi = [T_0T_1]/[T_1T_2] = [T_1T_2]/[T_2T_3] = [T_2T_0]/[T_1T_2] = ([T_2T_3]+[T_3T_0])/[T_3T_0] = [T_2T_3]/[T_3T_0] + 1.$

In terms of the ratio **φ,** of the long side to the short side of the Castor triangles, this equation becomes
(*) $\varphi = 1 / \varphi + 1.$

Thus to construct a decagon and pentagon we must find the point T_3 on the radius $[T_0T_2]$ that divides it in the ratio φ. The length $[T_0T_3] = [T_3T_1] = [T_1T_2]$ is the length of each side of the decagon. [1] T_3 is not hard to construct because (*) the defining algebraic relation for φ yields numerically $\varphi = (\sqrt{5} + 1)/2$ and $1/\varphi = (\sqrt{5} -1)/2$ and also dividing (*) by φ yields $1 = 1/\varphi^2 + 1/\varphi$.

To see how Figure 2 can be constructed, given a circle *C* with center T_0 and radius $[T_0,T_2]$ as in Figure 3. We take the radius as our unit of length; $[T_0T_2] = 1$. Bisect $[T_0T_2]$ and label the midpoint M. Construct A_1 satisfying $[T_2A_1]$ perpendicular to $[T_0,T_2]$ and length $[T_2A_1] = [T_2M] = ½$. Then $[T_0,T_2,A_1]$ is a right triangle with hypotenuse length $[A_1T_0] = \sqrt{5}/2$. Let A_2 be on the hypotenuse satisfying $[A_1A_2] = [A_1 T_2] = ½$. Then $[A_2T_0] = (\sqrt{5}/2 – 1/2) = 1/\varphi$.

Next find T_3 on $[T_0,T_2]$ satisfying $[T_0T_3] = [A_2T_0] = 1/\varphi$. Then $[T_3T_2] = 1 – 1/\varphi = 1/\varphi^2$. Hence $[T_0T_3]/[T_3T_2] = (1/\varphi)/(1/\varphi^2) = \varphi$. Finally, find T_1 on *C* satisfying $[T_0T_3] = [T_3T_1] = [T_1T_2]$ which is the side of the decagon. From which it and the pentagon can be constructed.

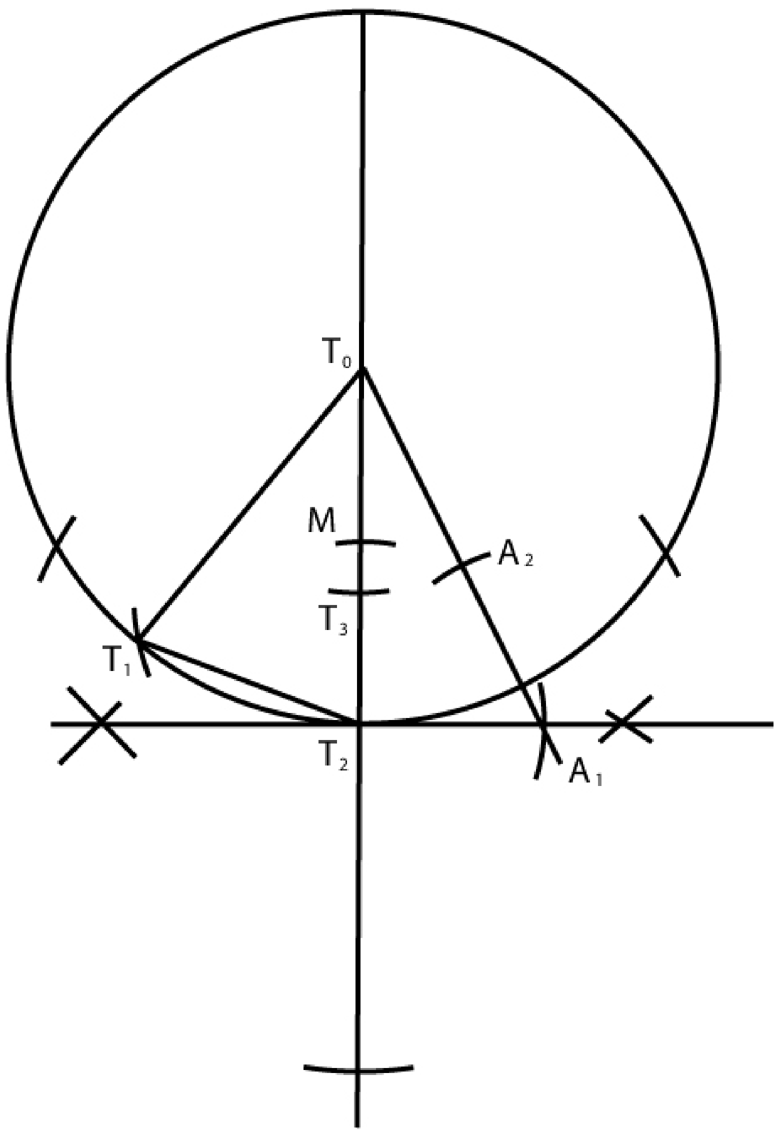

Figure 3: Construction diagram

Constructing the Quilt

Using this correct construction, we can now proceed with quilt design. We collaborated on several designs. They came up sequentially while trying to fix problems in each design. The first design was to illustrate all of the Gemini twins and mothers. It was not aesthetically pleasing, even though it did show all of the twin pairs. See Figure 4.

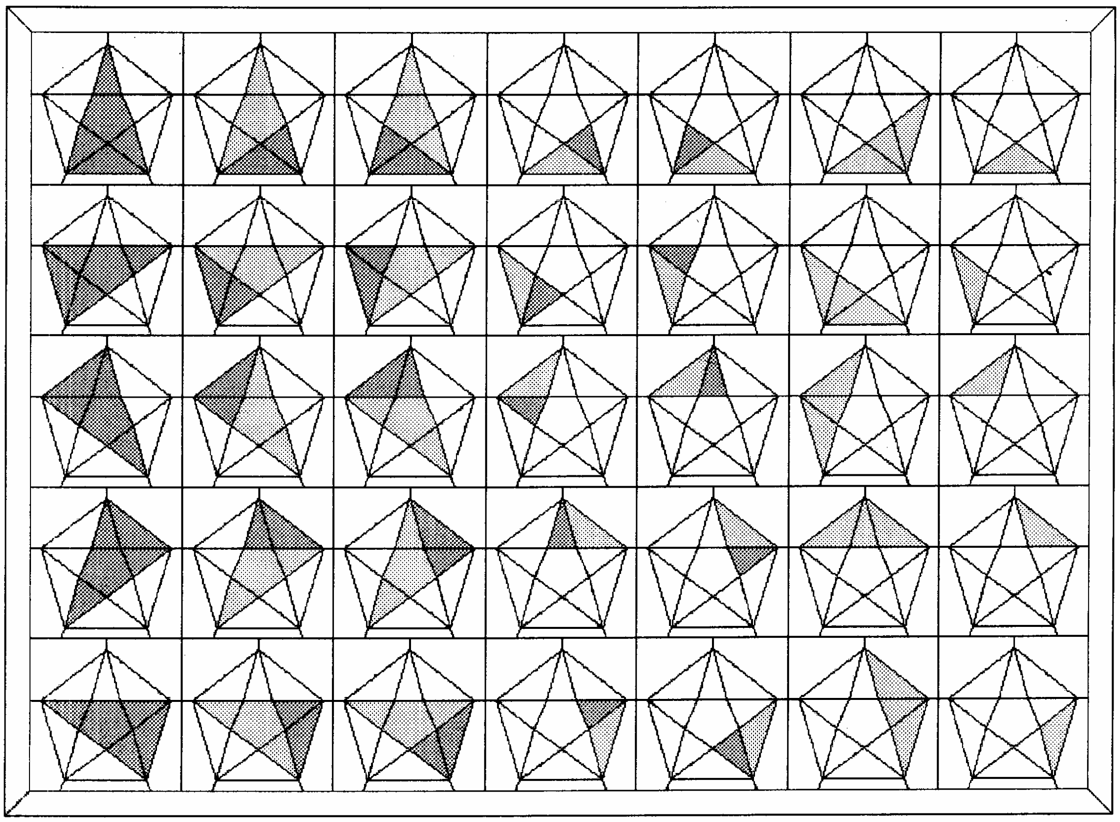

Figure 4: Electronic Quilt 4 Five point star design

The darker grey triangles are the Castors while the lighter grey triangles are the Pollux. This uses the misaligned block from EQ4 [2]. If you carefully lay a ruler along one side of the interior pentagon, you can see that it does not line up with the two Castor triangles that are the points of the star. Since the quilt program is supposed to print out the pattern pieces, I (Mary) would not use it for a real quilt. However, it is included because it shows all the triangles and orientations quite nicely.

It also demonstrates how to piece the block, since some of the interior lines extend to the side of the block, these are seam lines. There is more than one way to piece the block, but since it was unused, it is not necessary to say how.

I decided to inlay the four paired twins (columns 2 to 5 above) and the two solo Pollux (column 6 & 7 above) onto a larger Mother imbedded in a pentagon. Then by varying the orientation by rotation of the entire design one could see all the possible triangles. Figure 5.

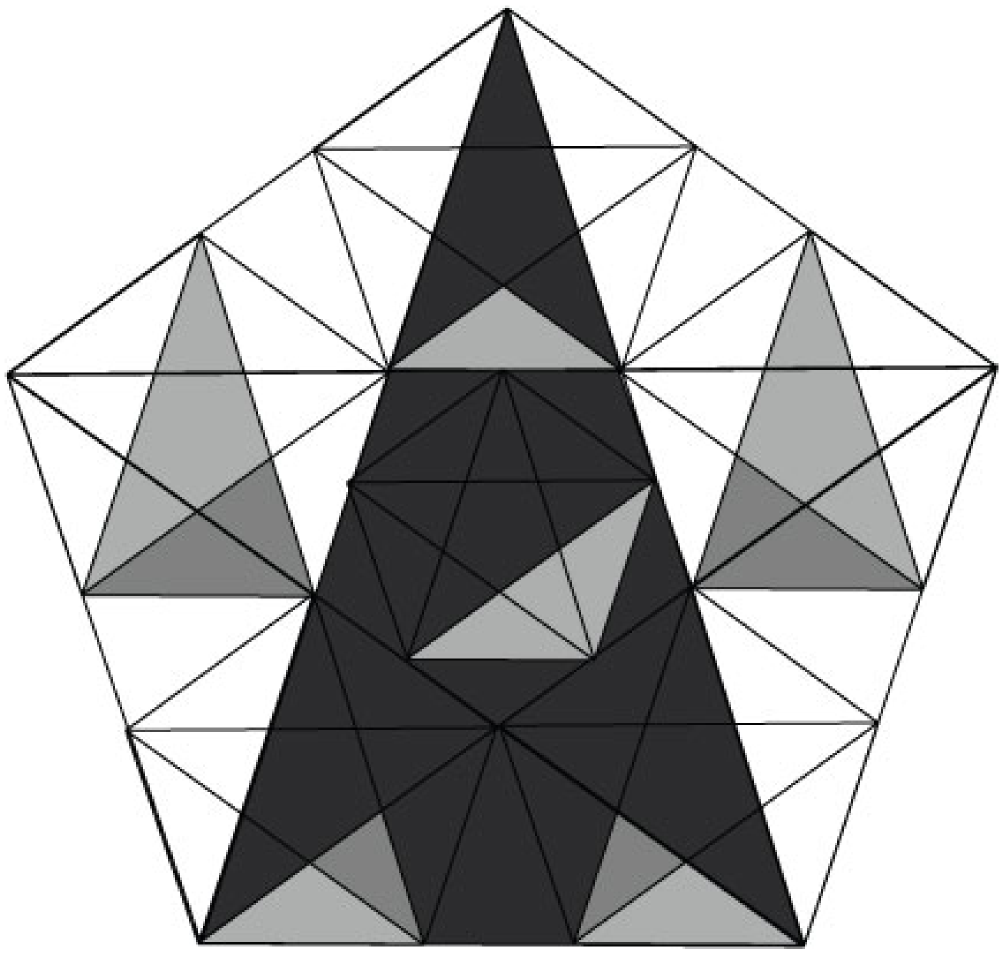

Figure 5: First design

When coloring the above design, I constantly lost the Mother Castor Triangle amid the colors when the largest Pollux and Castor pairs were in the lowest positions at the base of the Mother. I found that I needed two colors for the Castor triangles because if I had only one, how could it be seen against the larger Mother Castor? While it is not possible to have total symmetry in this design, a pleasant balance is desired, and we hope, achieved.

Seaming this design looked to be possible from the inside out, with modifications made to insure straight seams (highly desired by quilters). It is also possible to develop the design through coloring to emphasize the pentagrams and omit the twin aspects as given here. Or the pentagrams can be quilted and thus retained even when not colored.

So I further modified the design so the twin pairs and solo triangles are much smaller and finally all the triangles show up. However, this design would be very tedious to sew. Figure 6.

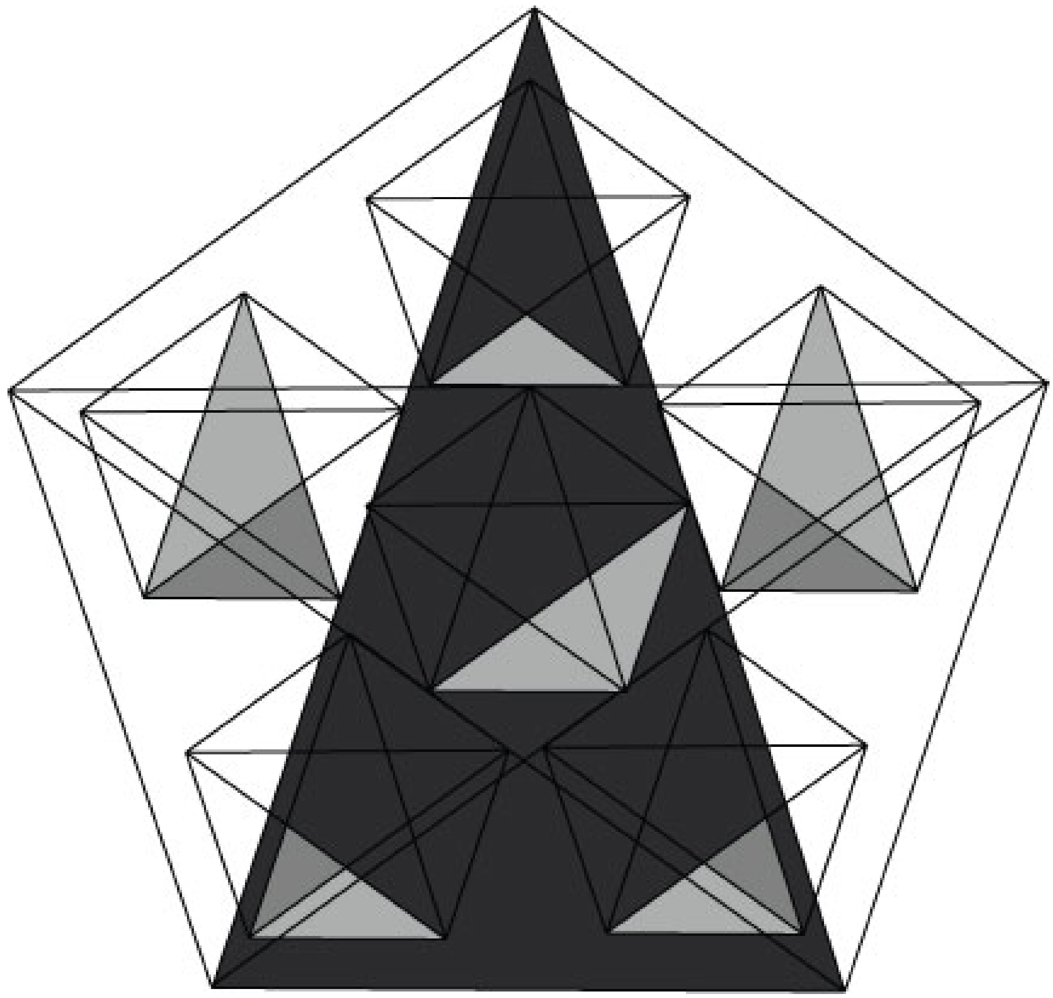

Figure 6: Adjustment of interior pentagons to be all one size.

While the main seam lines are retained from Figure 5, there would be more quilt lines to quilt and some confusion would result as questions could arise concerning design lines versus the ‘is that a seam line or a quilt line or did you goof’. Then there are some nasty places at the corner of each one of the outer ring of pentagons where it approaches the corner of the outer pentagon. There is this $2\pi/5$ angle with the $8\pi/5$ angle that quilters just don’t want to sew. It is possible to add a seam line to change the angles to a straight seam, but that could detract form the overall design. The problem of design line that cannot be incorporated as a seam line is an ongoing challenge for mathematical quilts.

But one last idea occurred to me of situating them as the seven sisters, of six hexagons around a central hexagon, but after consultation, we finally decided on the second design (Figure 5) with a twist. Figure 7.

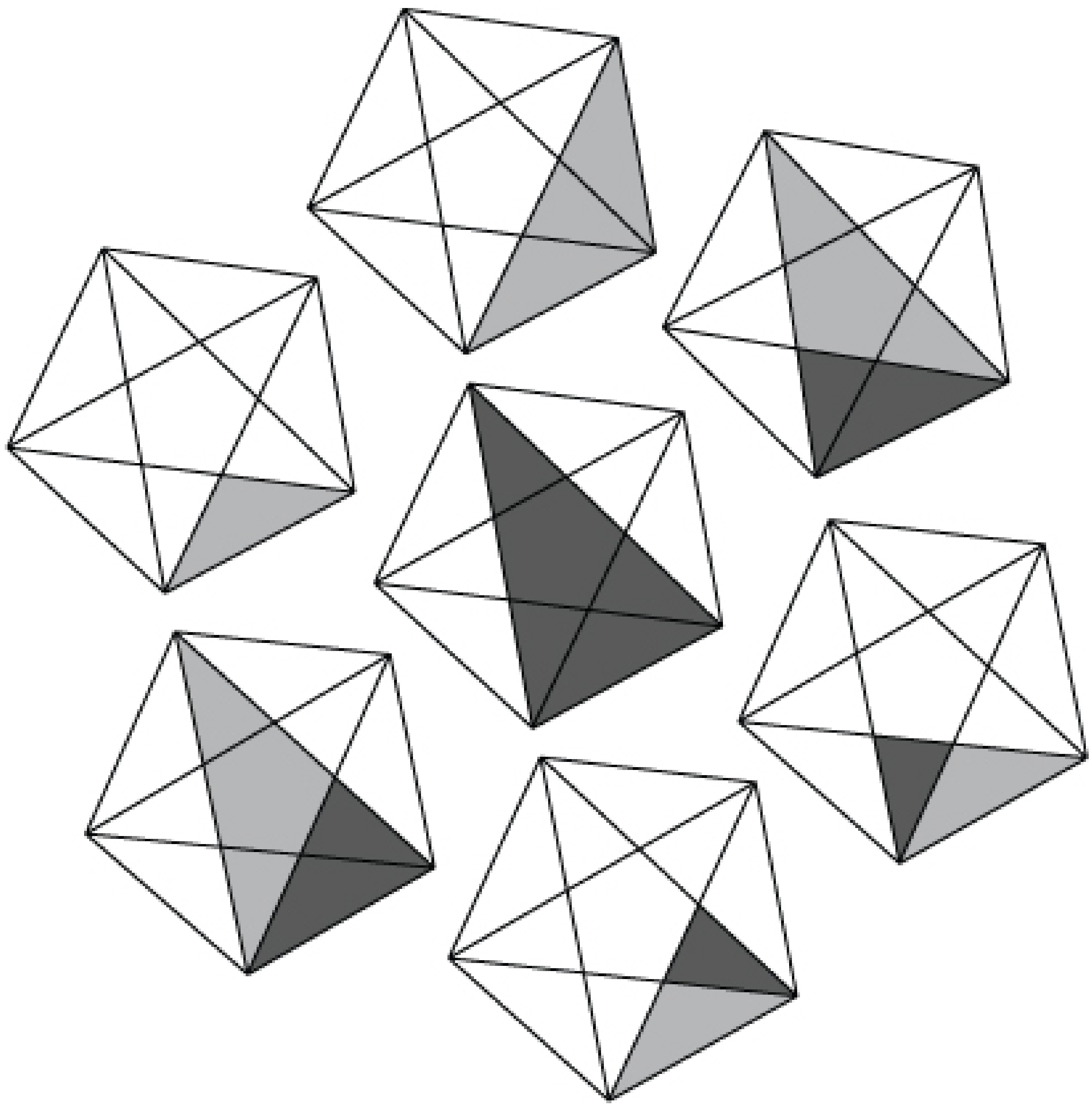

Figure 7: Seven Sisters variation of design

This clearly shows all of the twin pairs and solos, and can be rotated to show the proper orientation. The problems sewing here are the fact of sewing a pentagon within a hexagon. One could just sew fabric of the background color and then fit them with out a pattern for the background pieces. An alternative would be to construct the seven pentagons/pentagrams and suspend them with monofilament as a mobile. If this proves successful, we will show it off later.

References

[1] Euclid, Book IV Proposition 10 and 11 translations by Sir Thomas L. Heath
[2] Electronic Quilt 4, 1999
[3] L. Gordon Plummer, *The Mathematics of the Cosmic Mind* page 186, 1970

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