Medieval Islamic Architecture, Quasicrystals, and Penrose and Girih Tiles: Questions from the Classroom

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Abstract

Tiling Theory studies how one might cover the plane with various shapes. Medieval Islamic artisans developed intricate geometric tilings to decorate their mosques, mausoleums, and shrines. Some of these patterns, called girih tilings, first appeared in the 12th Century AD. Recent investigations show these medieval tilings contain symmetries similar to those found in aperiodic Penrose tilings first investigated in the West in the 1970's. These intriguing discoveries may suggest that the mathematical understanding of these artisans was much deeper than originally thought.

Connections like these, made across the centuries, provide a wonderful opportunity for students to discover the beauty of Islamic architecture in a mathematical and historical context. This paper describes several geometric constructions for Islamic tilings for use in the classroom along with projects involving girih tiles.   pen questions, observations, and conjectures raised in seminars across the United Arab Emirates are described including what the medieval artisans may have known as well as how girih tiles might have been used as tools in the actual construction of intricate patterns.

1. Islamic Tilings and Traditional Strapwork

The Islamic world has a rich heritage of incorporating geometry in the construction of intricate designs that appear on architecture and tile walkways as well as patterns on fabric, see [4]. This highly stylized form of art has evolved over the centuries from simple designs to fairly complex geometry involving a high degree of mathematical symmetry. Many of these complex designs can be constructed using a "strapwork method" where circles and squares are transformed into stars and overlapping lattices to form a more intricate symmetric pattern (Figure 1). The Alhambra Palace, see [6], the 15th Century Moorish architectural wonder in Granada, Spain contains many excellent examples of these Islamic constructions (Figure 2).

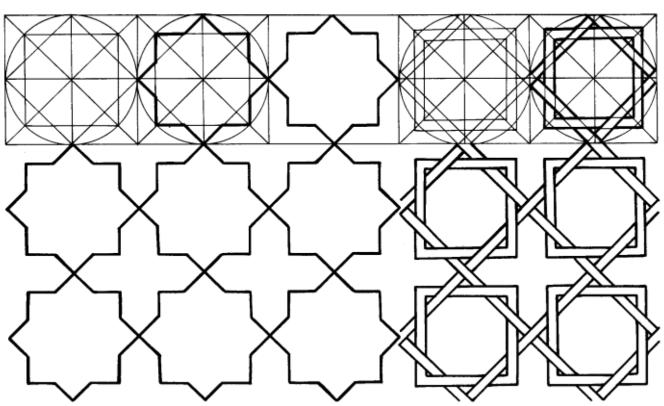

Figure 1: *Strapwork Method Showing Construction from Circles to Lines to Stars to Overlapping Lattices, Geometer's Sketchpad*

Figure 2: *Alhambra Tiling, Photo by R. Tennant*

The vast majority of these intricate patterns repeat in a periodic manner as the following two tilings from the Alhambra illustrate. Some patterns eminate from a central point and maintain periodic symmetry on radial axes (Figure 3).   ther patterns repeat perfectly in two linearly independent directions (Figure 4) and are referred to as the two-dimensional crystallographic groups.

Figure 3: *Alhambra Tiling – Radial Symmetry, Photo by R. Tennant*

Figure 4: *Alhambra Tiling – Periodic by Translations, Photo by R. Tennant*

Although many of the patterns found on Islamic architecture can be constructed using periodic methods like strapwork with straightedge and compass, see [5], there are numerous examples which appear to be nonperiodic and contain symmetries which may require additional construction techniques. The tilings below from 15^{th} Century Turkey (Figure 5) and from 17^{th} Century India (Figure 6) illustrate decagonal (ten-point) symmetry which in modern times has been discovered in quasi-crystal structures. Recent discoveries, by physicist Peter Lu of Harvard University, suggest that the medieval artisans who created these patterns had a deeper understanding of geometry than originally thought, see [2].

Figure 5: *Sultan's Lodge, Ottoman Green Mosque in Busra, Turkey (1424 AD), Photo by W. B. Denny, see [3]*

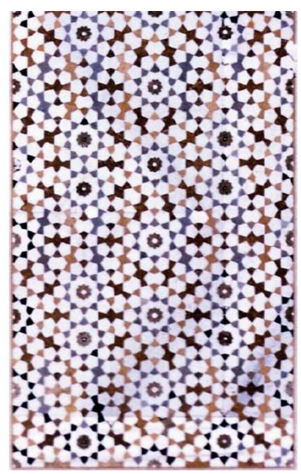

Figure 6: *Mausoleum of I'timad al-Daula in Agra, India (1622 AD), Photo by M.W. Meister, see [3]*

2. Nonperiodic and Aperiodic Tilings

Patterns that do not repeat in a linear direction are called nonperiodic. The tiling by Heinz   oderberg (Figure 7) from the 1930's exhibits a type of spiral symmetry. The pattern referred to by its sphinx-like tiles (Figure 8) has a self-similar fractal quality as each sphinx may be dissected into four smaller sphinxes.

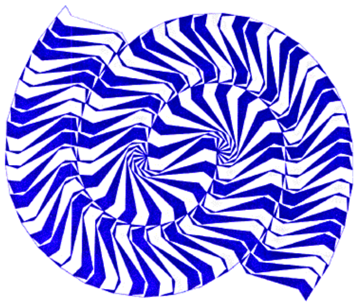

Figure 7: *Voderberg Spiral Tiling*

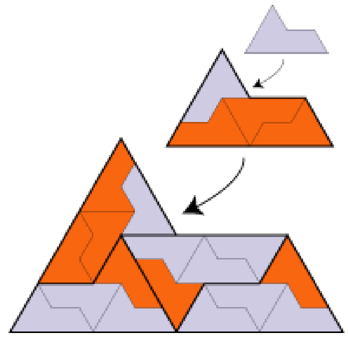

Figure 8: *Sphinx Tiling – Expands by Four*

In the 1970's, new tilings were discovered that not only were nonperiodic but could not be rearranged to be periodic. An example of this type of "aperiodic" tiling was discovered by Roger Penrose and consisted of two rhombuses (Figure 9). The gluing instructions for these "Penrose rhombs" were determined by arcs which form a nonperiodic pattern (Figure 10). Although this pattern is not periodic, it is highly structured and contains quasi-periodic five-fold rotational symmetry.

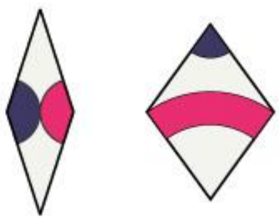

Figure 9: *Penrose Rhombs*

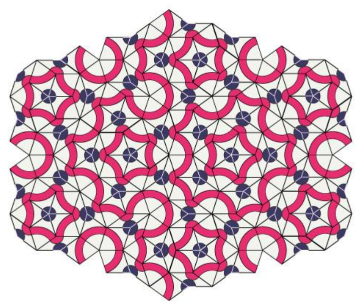

Figure 10: *Penrose Aperiodic Tiling*

Aperiodic symmetry has been incorporated into the design of modern architecture. Storey Hall (Figures 11-12) at the Royal Melbourne Institute of Technology (RMIT) was constructed in the 1990's utilizing symmetry based on Penrose's aperiodic rhombs. The innovative design and creative blending of the hall into the surrounding 19th Century Melbourne neighborhood has won several architectural awards for this modern structure.

Figure 11: *RMIT Storey Hall,*
Melbourne, Australia,
Photo by Tim Griffith

Figure 12: *Interior Auditorium – RMIT Storey Hall*
Melbourne, Australia,
Photo from www.a-r-m.com.au/

3. Medieval Tilings with Girih Tiles

Examples of intricate nonperiodic tilings dating from the 10th to 15th Century AD may be found throughout the world. The Darb-i Imam Shrine (1453 AD) in Isfahan, Iran (Figures 13-14) provides excellent examples of this type of ornamentation. Recent discoveries, see Lu and Steinhardt [2], have provided intriguing insights into how the craftsmen may have assembled the tilings at the shrine in a manner that maintained the intricate symmetry of nonperiodic quasi-crystals.

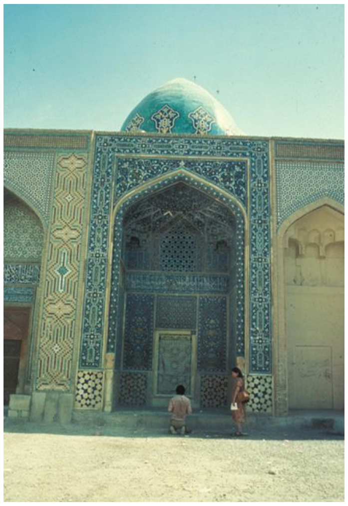

Figure 13: *Portal, Darb-i Imam Shrine, Isfahan, Iran, Photo by S. Blair and J. Bloom*

Figure 14: *Mosaic Faience Work, Darb-i Imam Shrine, Isfahan, Iran, Photo by S. Blair and J. Bloom*

A set of girih tiles (Figure 15) consisting of a decagon, a pentagon, a hexagon, a bowtie, and a rhombus provide detailed instructions for creating complex patterns. The girih tiles themselves are not part of the final pattern but rather the line decoration on the girih tiles determine the design. The girih tiles might be thought of as templates that determine the placement of the actual tiles. A reconstruction of the process of transformation from the girih tiles to the architectural design is illustrated on the 15th Century Timurid Shrine (Figure 16). The spandrel tiling from 13th Century Iraq (Figure 17) is shown along side the associated girih tile pattern.

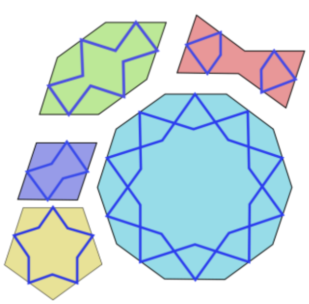

Figure 15: *Five Types of Girih Tiles, see [2], Drawing by Peter J. Lu*

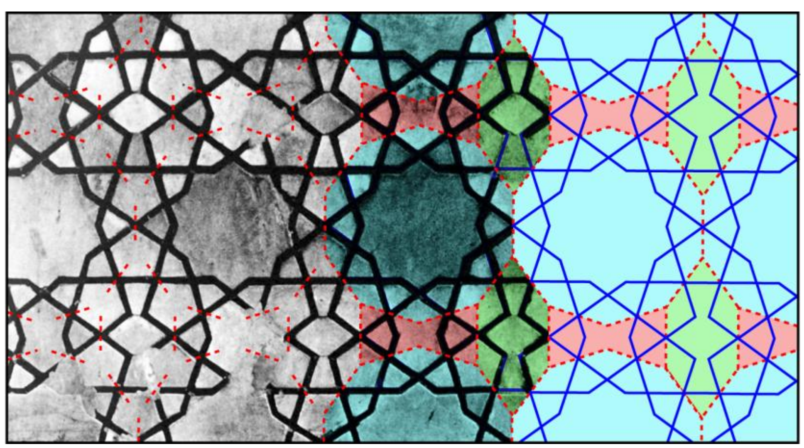

Figure 16: *Periodic Tiling with Actual Tiling (left) Transformed to Girih Tiling (right), Timurid Shrine of Khwaja Abdullah Ansari at Gazargah in Herat, Afghanistan (1425-1429 AD), see [2], Modified from Figure 1E*

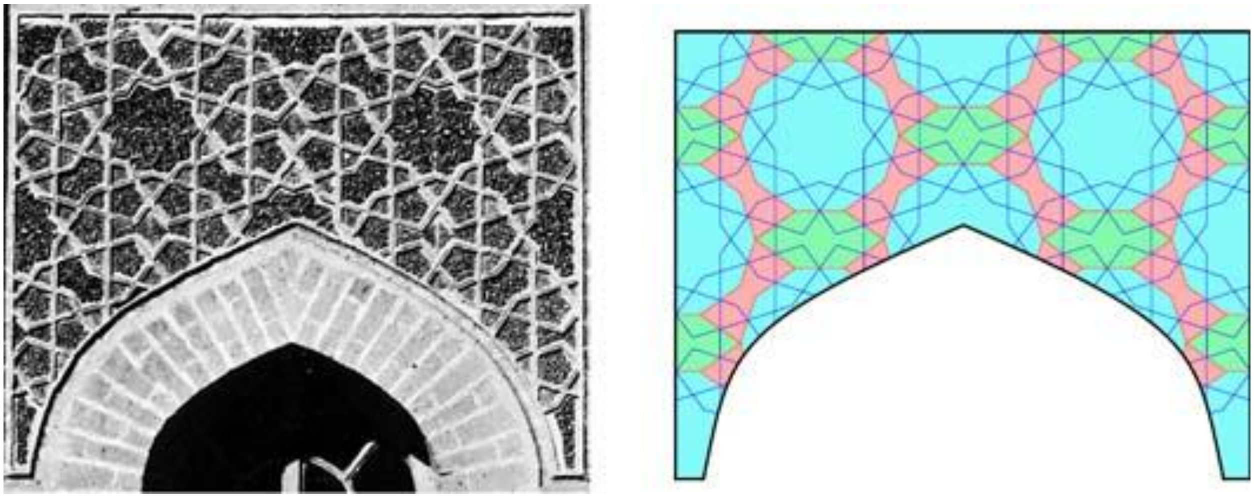

Figure 17: *Spandrel from the Abbasid Al-Mustansiriyya Madrasa, Baghdad, Iraq (1227-34 AD, see [3] (left), and with Girih Tiles (right), Drawing by Peter J. Lu*

4. Conjectures from the Classroom

The girih tilings provide an intriguing bridge from Islamic architecture of the medieval age to the modern era and 20th Century Penrose aperiodic tilings. This interesting historical and cultural connection provides a basis for further research and discussions by students studying modern geometry and group theory, as well as the history of mathematics. This crossroad of mathematics and cultural architecture may also be thought of as an interdisciplinary tool to be utilized by teachers in describing mathematics as an endeavor of the human spirit. Below are some open questions, conjectures, and observations that are the result of several discussions from seminars and classes given throughout the United Arab Emirates.

Question 1. *Where and when did the shift occur from "direct strapwork" to the "girih-tile paradigm?"*

There are several existing examples of complex decagonal tiles some of which date back to 1200 AD. These architectural patterns are found in a wide range of sites ranging from Turkey to India and from Iraq to Uzbekistan.

Question 2. *Many of the examples of medieval tilings with the quasicrystal patterns have defects but these defects are local and usually can be fixed by simple rearrangements or rotations of tiles. Are these defects most likely a mistake made by a tile craftsman?*

In their research on patterns from the Darb-i Imam Shrine in Isfahan, Iran, Lu and Steinhardt noted that a particular girih tiling matched a Penrose tiling of kites and darts almost exactly. In viewing the placement of all 3700 tiles on this 15th Century pattern (Figure 18), they found that there were 11 defect variations from the modern Penrose tiling (Figure 19), but by shifting pairs of girih tiles each defect can be removed. This would certainly be a strong argument that some defects in the pattern occurred either when the original tiles were placed or during reconstruction work. It should be noted that although this particular pattern of 3700 girih tiles on this "finite" spandrel may remarkably be transformed into a fragment of a Penrose tiling, there would not be perfect matching if both patterns were extended to infinity.

Figure 18: *Spandrel from Darb-i Imam Shrine, Isfahan, Iran (1453 AD) with Girih Tiles Overlaid, Photo by K. Dudley and M. Elliff*

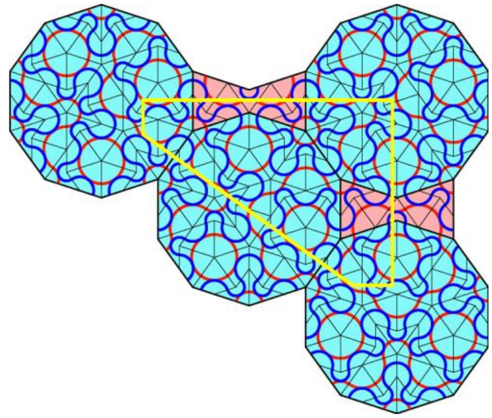

Figure 19: *Mapping to a Penrose Tiling Extended Beyond the Spandrel, see [2], Drawing by Peter J. Lu*

Question 3. *Is there any historic evidence that the girih tiles were used in the manner of templates to construct these intricate patterns?*

The Topkapi Scroll (Figure 20) is a 15th Century collection of architectural drawings created by master builders in the late medieval period in Iran. The scroll contains 114 individual geometric drawings detailing the theory and instructions for laying intricate patterns on walls and vaulted ceilings. Panel 50 of the scroll (Figure 21) is shown with girih tiles superimposed. The entire set of five girih tiles is shown on Panel 28 of the Topkapi Scroll (Figure 22).

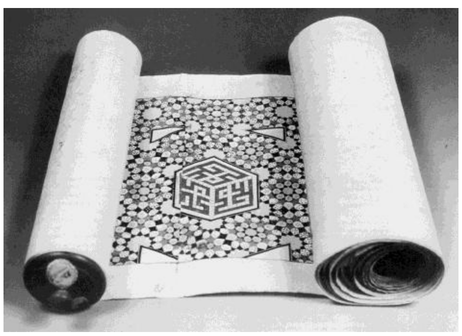

Figure 20: *Topkapi Scroll, Topkapi Museum, Istanbul, Turkey, Photo from http://www.ee.bilkent.edu.tr/~history/geometry.html*

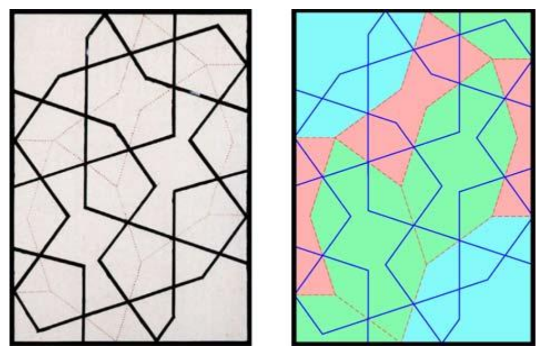

Figure 21: *Panel 50 from the Topkapi Scroll (left) with Superimposed Girih Tiles (right), see [2], Drawing by Peter J. Lu*

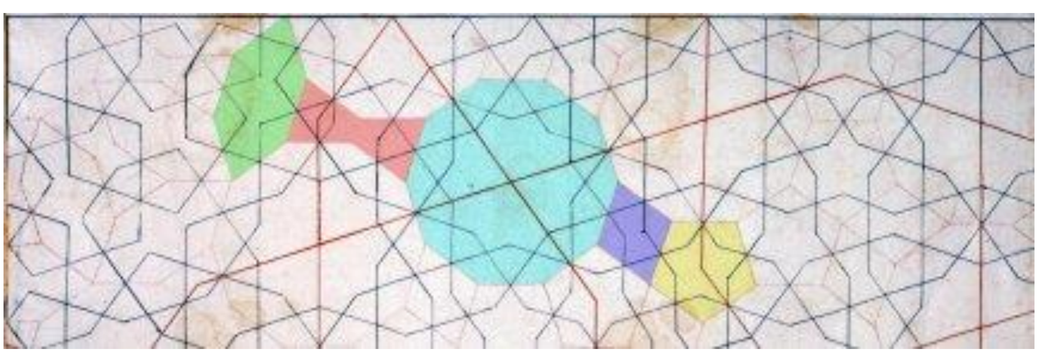

Figure 22: *Panel 28 of the Topkapi Scroll with the five Girih Tiles, see [2]*

Question 4. *Since any tile pattern on a building is finite and therefore only a fragment of the infinite plane, how can we know that these tilings are actually nonperiodic?*

The answer may lie in the method of dissection. A periodic tiling can easily be constructed by repeated translations of a generating tile or set of tiles. Another approach is to methodically dissect tiles to create a new tiling where the size of the tiles is now smaller. The simple case for tiling by squares would consist of starting with one square and dissecting into 4 squares and then dissecting each new square to form 16 squares and then 64 squares and so on. In theory the original square could be expanded in size to infinity as the dissection continued and the tiling by squares would cover the plane. Beside the five girih tiles shown in color, Panel 28 of the Topkapi Scroll (Figure 22) shows a larger version of the girih tiles highlighted by the faint red lines. The existence of two different sizes of girih tiles on the same scroll suggests knowledge by the medieval artisans of the method of dissection.

In order to determine if this method of dissection would produce a nonperiodic tiling when an architectural fragment was continued out to infinity, an analytic proof is necessary utilizing the dissection rule, see [2]. As an example, the portal from the 15th Century Darb-i Imam Shrine (Figure 23) in Isfahan, Iran is shown along with two different length scales (Figure 24) in order to determine the dissection rule.

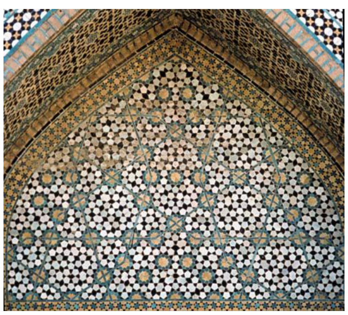

Figure 23: *Portal from the Darb-i Imam Shrine, Isfahan, Iran (1453 AD), Photo by K. Dudley and M. Elliff*

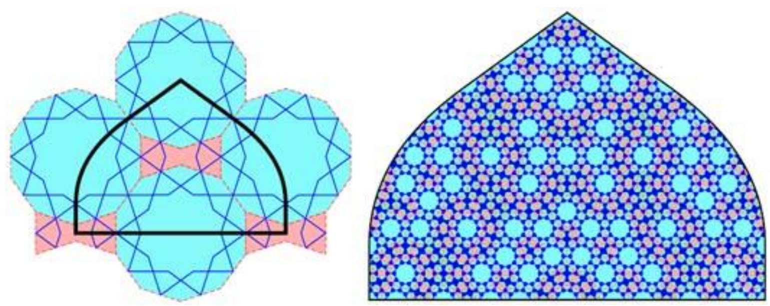

Figure 24: Same *Spandrel from the Darb-i Imam Shrine, Showing Two Successive Generations of Girih Tiles, Drawings by Peter J. Lu*

The dissection rule, see [2] for each girih tile can be described in terms of how many smaller tiles result when the larger tiles are dissected. For the case of tilings consisting of decagons, bowties, and hexagons, the following dissections can be determined by counting tiles (Figure 25).

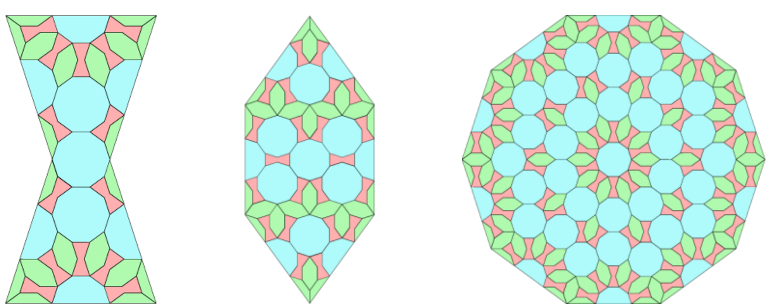

Figure 25: *Dissection Rule for Bowtie, Hexagon, and Decagon, Drawings by Peter J. Lu*

1 LARGE B WTIE = 14 small decagons + 14 small bowties + 6 small hexagons
1 LARGE HE AG N = 22 small decagons + 22 small bowties + 10 small hexagons
1 LARGE DECAG N = 80 small decagons + 80 small bowties + 36 small hexagons

This overall dissection rule can be written in the form of a transformation matrix.

$$\begin{bmatrix} 14 & 22 & 80 \\ 14 & 22 & 80 \\ 6 & 10 & 36 \end{bmatrix} \begin{bmatrix} \textit{LARGE BOWTIES} \\ \textit{LARGE HEXAGONS} \\ \textit{LARGE DECAGONS} \end{bmatrix} = \begin{bmatrix} \textit{small bowties} \\ \textit{small hexagons} \\ \textit{small decagons} \end{bmatrix}$$

In order to determine if this method of dissection leads to a periodic tiling, the eigenvalues of the transformation matrix are calculated.

$\lambda_1 = 36 + 16\sqrt{5} = 4\varphi^6 \approx 71.78$, where $\varphi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio. The occurrence of the golden rational should not be surprising due to the pentagonal and decagonal symmetry of the pattern.

$\lambda_2 = 36 - 16\sqrt{5} \approx 0.22$

$\lambda_3 = 0$

Since the eigenvalue $\lambda_1 = 4\varphi^6$ is irrational this dissection rule will not result in a periodic tiling when carried out to infinity and so this pattern will be nonperiodic.

5. Conclusion

The recent discoveries linking the medieval world of Islamic tilings with the modern world of mathematical theory provide an interesting historical and cultural connection for further faculty and student research projects. As a classroom tool, this intriguing history provides motivation to increase student interest and excitement in mathematics, particularly, for students who share this history and culture. In the future, new discoveries may continue to unlock the mystery of how these medieval artisans developed and designed these beautifully intricate nonperiodic patterns and more may be learned about their true level of mathematical sophistication and understanding.

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