The Brachistochrone Problem, between Euclidean and Hyperbolic

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Abstract

We investigate discrete models of the upper-half plane endowed with various conformal metrics, which in essence are intermediaries between the standard Euclidean and the hyperbolic ones. The brachistochrone problem is related to a metric associated to arithmetic sequences.

1. Introduction

The transformations of the Euclidean plane based on discrete, translation preserving, 'wallpaper' groups are well-known in both math and art. Less famous are various tilings of the hyperbolic plane, which is the most famous model of a non-Euclidean geometry. In this work, we present a way of unifying the two geometries in the half-plane models via a parameterized family of conformal geometries.

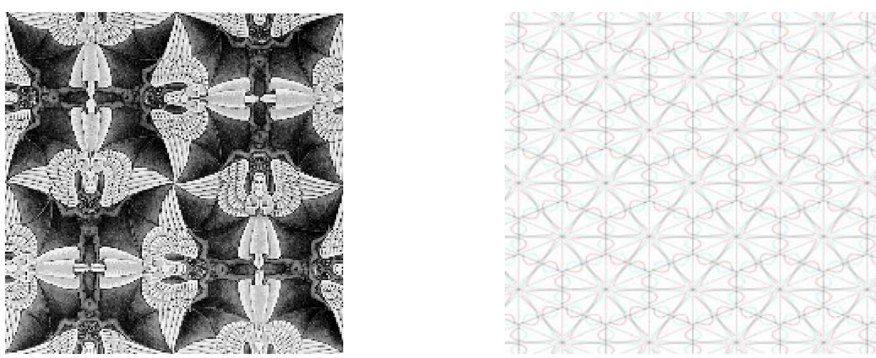
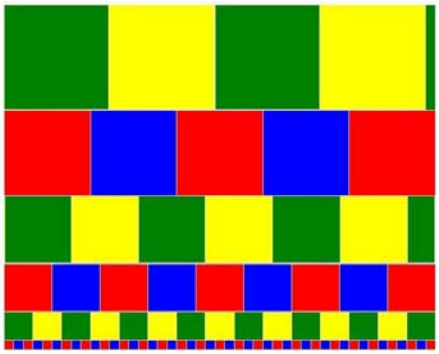


Figure 1: *Two patterns with different rotational symmetries*

2. Sequence Description of Discrete Conformal Geometries

To each sequence $\{a_1, a_2, a_3 \ldots\}$ of non-decreasing integers, one can associate a geometry of squares covering the half-plane, according to the following rules: First we fix the x-axis and origin, and we position the so-called cornerstone, a square with side length a_1, with its lower left vertex at the origin and sides parallel to the axes. Next we lay squares of side length a_1, adjacent to each other, and with the x-axis touching their lower horizontal side. We place the next layer of squares of side lengths a_2 above the previous layer, so that they touch the squares below and with one of the squares having its lower left vertex on the y-axis. The n^{th} layer is made up of squares of side length a_n with lateral sides touching each other and lower horizontal side touching the squares of length a_{n-1} below. One of the squares in each layer has a side on the y-axis. One extends this construction to the half-plane by reflection in the y-axis.

Example 1: Starting with the arithmetic sequence {1, 3, 5, 7, 9 …}, one obtains the geometry seen in Figure 2, which is related to the brachistochrone geometry as described in Section 5.



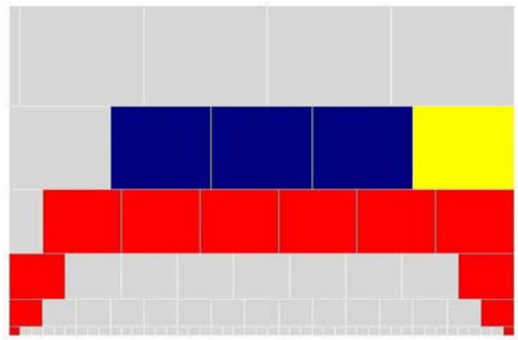
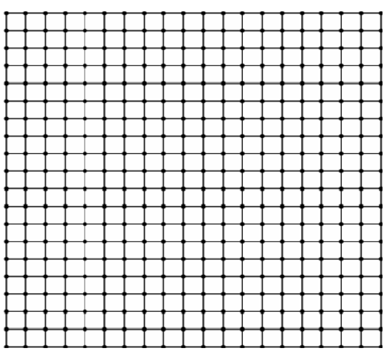


Figure 2a: *A discrete brachistochrone geometry* **Figure 2b**: A geodesic in the geometry

In other words, a square of size one is placed at the origin, represented here as the lower left of the figure. Next, squares of size one are laid horizontally. Above the first three squares is a square of side length 3, and one positions others of the same length at this level. Above the squares of length 3 are placed squares of side length 5 and so on.

Examples 2 and 3: The discrete square geometries of the constant sequence {1, 1, 1 …}, and the geometric sequence {1, 2, 4, 8 ...} is shown in Figure 3a and 3b, respectively.



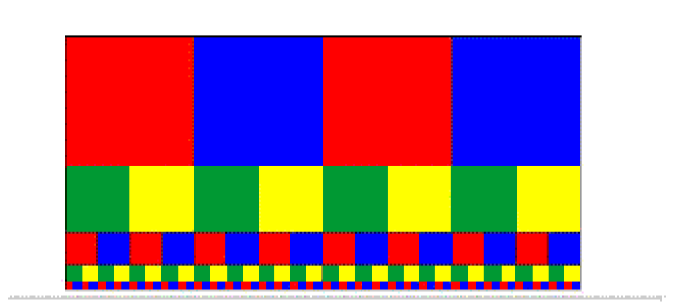
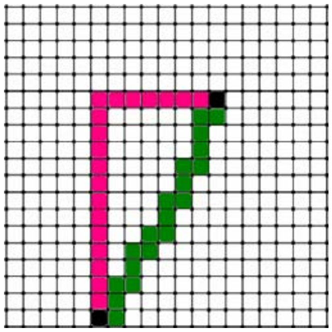


Figure 3a: *{1, 1, 1 ...}* **Figure 3b**: *{1, 2, 4, 8 …}*

Once a covering of the half-space is fixed, a geometry can be achieved by taking the squares as points, and defining the distances between any two different squares in the half-space as 1 plus the minimum number of squares (of any size) between them.

The discrete geometry associated to {1, 1, 1 …} seen in Figure 3a is very similar to the standard Euclidean geometry; the distance is called the taxicab metric. In the continuous plane it is up to a multiple (the square root of two) comparable with the standard distance using the Pythagorean Theorem. In Figure 4, the taxicab distance of the two darker squares is 20, and it is achieved for many different paths. A comparable Pythagorean distance is $\sqrt{7^2+13^2}$ which is less than 20 but larger than $20/\sqrt{2}$, so that the two distances are equivalent.



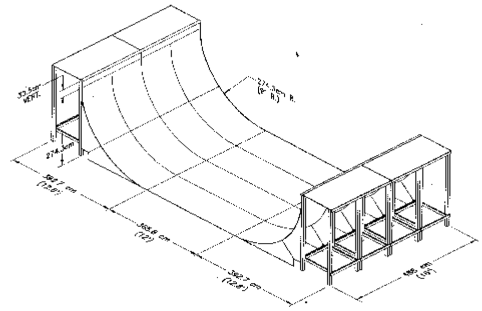
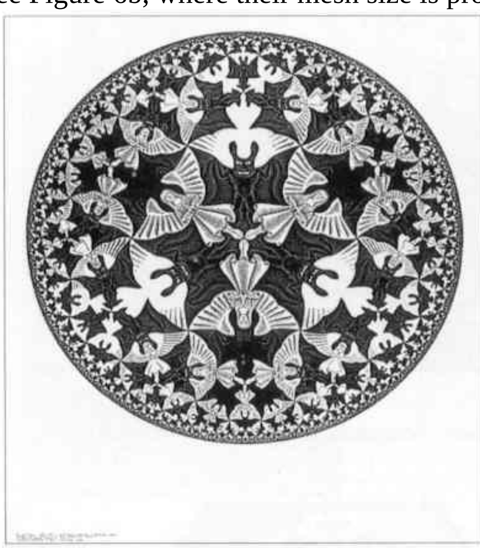


Figure 4: *Two geodesics for the taxicab metric* **Figure 5**: *A model of a skateboard ramp, [3]*

The geometry of the sequence {1, 2, 4, 8 …}, shown in Figure 3b, while initially difficult to understand, corresponds in fact to hyperbolic geometry. By this, we mean that for two squares reasonably far apart the chain of squares connecting them will follow a circular arc perpendicular to the x-axis [5]. On the other hand, if we look in Figure 2 to a chain connecting two squares at some distance, while initially it might seem as if this too would look roughly like a circular arc, it is in fact closer to a path traced out on a Spirograph™ on a line. This path, known as a cycloid, is described parametrically by $x = r(t - \sin t), y = r(1 - \cos t)$, where r is a fixed radius and t is the parameter. A cycloid is a reflection of the brachistochrone (*brachistos*="shortest" and *chronos*="time"), which is the shortest time-path between two points followed by an object falling under gravity, e.g. the shape of the skateboard ramp in Figure 5, [3]. In sections 4 and 5 the equations and geometry for the brachistochrone curve are given.

3. Discrete Conformal Geometries in Art and Nature

There are a boundless number of situations where discrete versions of the Euclidean geometry appear. Well-known examples are the wallpaper patterns in art, which illustrate the crystallographic plane groups and have numerous applications in physical chemistry. Similarly, the discrete hyperbolic geometry makes numerous appearances ranging from the art of Escher (Figure 6a) to models of polymers at an interface; see Figure 6b, where their mesh size is proportional to their distance to the boundary (slanted lines) [5].



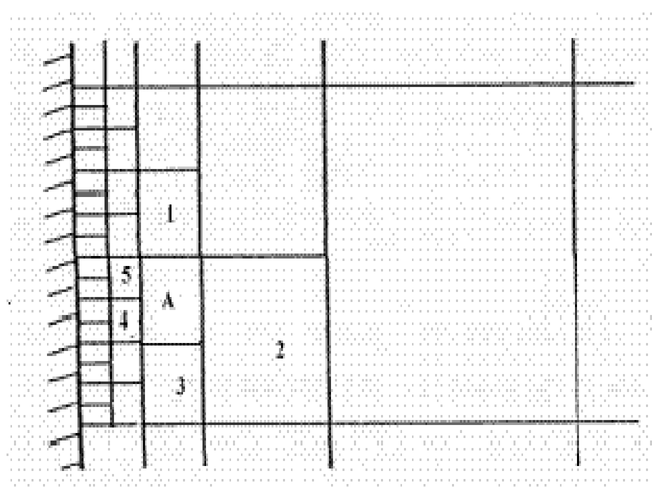


Figure 6a: *Circle Limit- Escher* **Figure 6b**: *Absorbed Polymer Layer- deGennes, [1]*

It is perhaps more astonishing the fact that a non-Euclidean non-hyperbolic locally conformal geometry for a model of the neocortex has been put forward as a way to explain some of its remarkable properties. In *On Intelligence*, Jeff Hawkins [4] describes the ability of the brain to quickly process information by passing it up through a Bayesian hierarchy pictured in Figure 7.

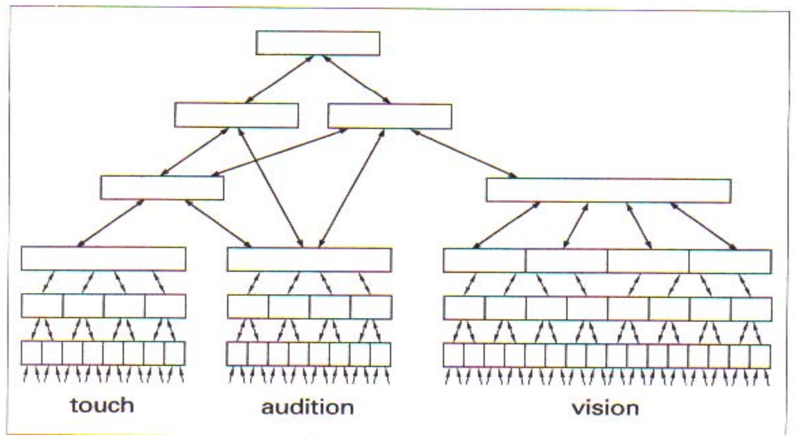


Figure 7: *A simplified model of the neocortex, [4]*

The ability of messages to pass quickly from one neuron to an enormous number of neighbors in a few steps is a critical property of the neocortex. One can see that, as in the polymer model in Figure 6b, the number of connections in the neuronal geometry grows faster than polynomially, this faster than polynomial growth is a hallmark of non-Euclidean geometry.

4. Continuous Conformal Geometries via Snell's Law

Imagine driving a vehicle from point P to Q in Figure 8. To the left of the interface one is on dry land and can proceed at a velocity v_1 =50 km/hr, while to the right of the interface one can only go to v_2 =20 km/hr. An interesting question to ask is which point O minimizes the travel time between P and Q. In fact, while the lateral distances x1 and x2, and the total distance y=y1+y2 are fixed, there is a freedom to choose y1 to minimize the travel time.

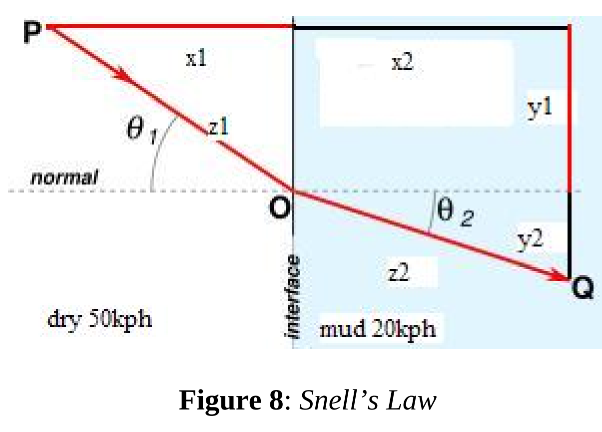


Figure 8: *Snell's Law*

The total travel time can be written as T = $\frac{z1}{50}+\frac{z2}{20}$, and using the Pythagorean Theorem one can rewrite it as a function of y1: $T(y1)=\frac{\sqrt{x1^2+y1^2}}{50}+\frac{\sqrt{x2^2+(y-y1)^2}}{20}$. The minimum is found to occur when $\frac{y1}{50\sqrt{x1^2+y1^2}}=\frac{y-y1}{20\sqrt{x2^2+(y-y1)}}$, which geometrically reads as $\frac{\sin(\theta_1)}{v_1}=\frac{\sin(\theta_2)}{v_2}$. This property, that is the invariance of the sine over the velocity, in optics is known as Snell's Law.

The relationship between Snell's Law and shortest paths in analytic geometry was discovered by Johann Bernoulli. Bernoulli imagined a medium with continuously varying optical properties, and found the correct curves using Snell's Law together with the relationship between the 'velocity' in a layer and the total distance of the layer from the boundary. If we look at Example 1 (Figure2) with this idea in mind, we see that there the top of the n^{th} layer is at distance 1+3+5+7+…+ (2n-1) = n^2 from the boundary, that is the x-axis. This corresponds to a falling object, when it has moved y^2 meters, having a velocity proportional to y i.e. $v^2(y)=2\cdot g\cdot y$.

With h the maximum height of the cycloid in Figure 10, Snell's Law and the approximation of sine in Figure 9 yield $\frac{\sin(\theta)}{v}=\frac{\sin(\theta)}{\sqrt{2\cdot g\cdot y}}=\frac{1}{\sqrt{2\cdot g\cdot h}}$ alias $\sin(\theta)\approx\frac{dx}{\sqrt{(dx)^2+(dy)}^2}=\sqrt{\frac{y}{h}}$. The last equality leads to a differential equation that can be solved explicitly, therefore the geodesics can be explicitly found.

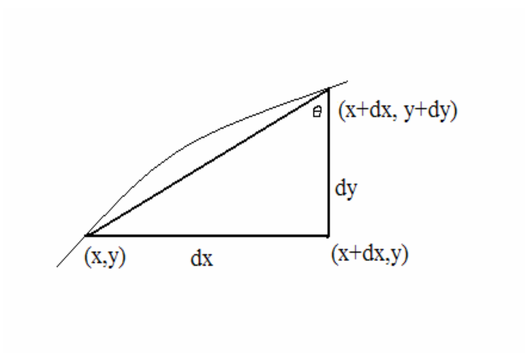


Figure 9: *Approximation of Sine*

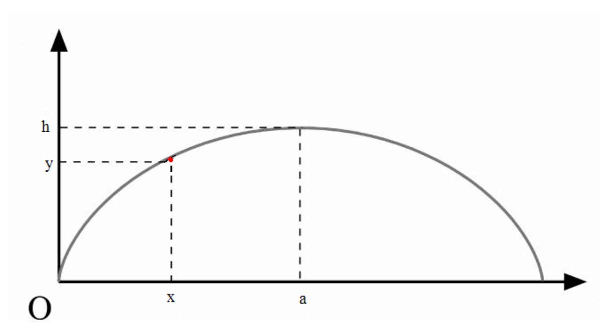


Figure 10: *The cycloid or brachistochrone*

5. A Parameterized Family of Conformal Geometries

In the language of differential geometry, a conformal metric on the upper half space $H=\{(x,y)\mid y>0\}$ is a rescaling of the regular Euclidean metric, symbolized by $ds^2=a(x,y)(dx^2+dy^2)$. For hyperbolic space the length of paths is measured via the metric $ds^2=y^{-2}(dx^2+dy^2)$, hence for example a vertical segment connecting (0, a) to (0, b) has hyperbolic length $\int_a^b y^{-1}\,dy$, as $dx^2=0$ along vertical lines. The logarithm obtained solving the integral indicates that this continuous geometry corresponds to a discrete geometry such as the one generated by the geometric series {1, 2, 4, 8…}, whose terms are powers of two. Similarly, a metric for the brachistochrone geometry is $ds^2 = y^{-1}(dx^2 + dy^2)$, and lengths have a factor of $y^{-0.5}$ coming from the square of the reciprocal of velocity, as seen in Section 4. We can see then the family of metrics $ds^2 = y^{-2a}(dx^2 + dy^2)$, with a>0, will interpolate between the Euclidean case, a=0, and the hyperbolic one, a=1. We mention that the brachistochrone case corresponds to a=0.5. Figure 11 shows a transformation from the Euclidean to the hyperbolic plane, via the brachistochrone geometry.

6. Conclusion

In this paper, we explore a way to build geometries from an arbitrary set of non-decreasing integers. Some of the traditional integer patterns correspond to models in the arts and sciences, where layers are built up to maximize given properties in situations of non-homogeneity. We discussed the connection between these discrete geometries and their continuous analogs. We finally provide a one-parameter family of conformal geometries for which the Euclidean and the hyperbolic geometries are extreme cases.

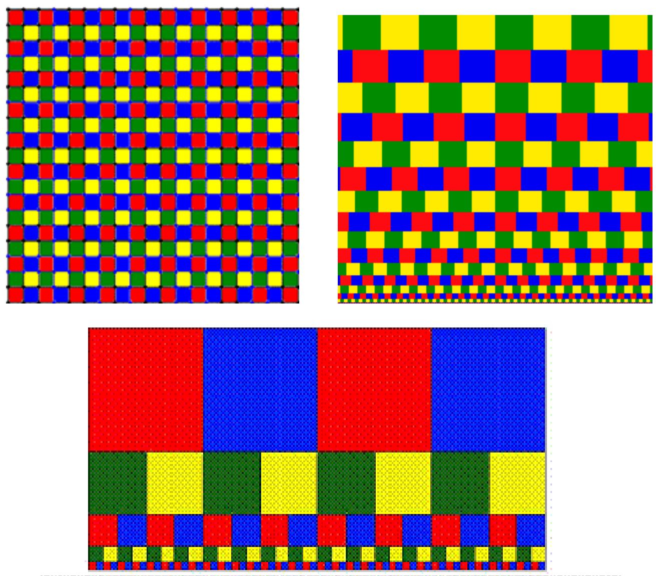


Figure 11: *From Euclidean to brachistochrone to hyperbolic*

References

[1] P.-G. de Gennes, *Simple Views on Condensed Matter*. World Scientific Publishing Company, 1998.

[2] P. Dombrowski, *The Brachistochrone Problem: a Problem of Elementary Differential Geometry*, Geometry and topology of submanifolds, VIII pp. 148-167 World Scientific Publishing Co. 1996

[3] R. Ferreol and J. Mandonnet, *Courbe Brachistochrone* at http://www.mathcurve.com/

[4] J. Hawkins, *On Intelligence*, Holt Paperbacks 2005

[5] R. Smits, *Square Decompositions with Hyperbolic Consequences in Art, Chemical Physics and Mathematics*, Bridges Proceedings 2003