Coloring Uniform Honeycombs

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Abstract

In this paper, we discuss a method of arriving at colored three-dimensional uniform honeycombs. In particular, we present the construction of perfect and semi-perfect colorings of the truncated and bitruncated cubic honeycombs. If G is the symmetry group of an uncolored honeycomb, a coloring of the honeycomb is *perfect* if the group H consisting of elements that permute the colors of the given coloring is G. If H is such that $[G : H] = 2$, we say that the coloring of the honeycomb is *semi-perfect*.

Background

In [7, 9, 12], a general framework has been presented for coloring planar patterns. Focus was given to the construction of perfect colorings of semi-regular tilings on the hyperbolic plane. In this work, we will extend the method of coloring two dimensional patterns to obtain colorings of three dimensional uniform honeycombs. There is limited literature on colorings of three-dimensional honeycombs. We see studies on colorings of polyhedra; for instance, in [17], a method of coloring shown is by cutting the polyhedra and laying it flat to produce a pattern on a two-dimensional plane. In this case, only the faces of the polyhedra are colored. In [6], enumeration problems on colored patterns on polyhedra are discussed and solutions are obtained by applying Burnside's counting theorem. The works [14, 19] highlight edge-colorings of the platonic solids. There are studies on colorings of three-dimensional space using an algorithm that makes use of the group structure of the Picard group [1, 2, 21]. Cross sections of the colored three-dimensional patterns were used to produce colored two-dimensional Euclidean patterns.

We find the occurrence of colored *honeycombs* (space filled with polyhedra) in different places; for instance, as representations of geometric constructions, or as models of chemical structures. Shown in Figures 3(a), 3(b) and 4(b) are illustrations of colored honeycombs. Interestingly, the colorings shown are representations of three different uniform constructions of the bitruncated cubic honeycomb, a honeycomb consisting of truncated octahedra. For example, the coloring with two colors in Figure 3(a) represents two types of truncated octahedra: half are obtained from the original cells of the cubic honeycomb and the other half are centered on vertices of the original honeycomb. In Figure 4(b), this colored honeycomb is referred to as the cantitruncated alternate cubic – there are 3 types of truncated octahedra in 2:1:1 ratios. In Figure 3(b) there are 4 types of octahedra in 1:1:1:1 ratios; each type is represented by a different color. In [18], a bitruncated honeycomb is used to represent a spongy graphite network of carbon atoms in 3-dimensional space.

In this paper, we present the construction of colored honeycombs where an entire cell gets one color.

In recent works [8, 16], a method for determining subgroups of three-dimensional symmetry groups in spherical, Euclidean or hyperbolic 3-space was discussed. The approach, based on concepts on color symmetry theory, allows for the characterization of each subgroup in terms of the symmetries it contains. This development is helpful in the construction of colorings of honeycombs especially in hyperbolic space, since the subgroup structure of hyperbolic symmetry groups is not widely known. As will be seen in this work, the subgroup structure of the symmetry group of a given honeycomb plays a significant role in arriving at colorings of the honeycomb.

Uniform honeycomb

We start the discussion by defining uniform honeycombs. A polyhedron is called *uniform* if its faces are regular polygons and it satisfies the property that its group of symmetries acts transitively on its vertices. A *uniform* honeycomb is a three dimensional honeycomb with uniform polyhedra as its cells and where the symmetry group of the honeycomb acts transitively on its vertices. Uniform honeycombs are also called *Archimedean* honeycombs.

In three-dimensional Euclidean space, twenty-eight such honeycombs exist: the cubic honeycomb and seven truncations thereof; the alternated cubic honeycomb and four truncations thereof; ten prismatic forms based on the uniform plane tilings (eleven if including the cubic honeycomb); and five modifications of some of the mentioned by elongation or gyration [11, 13]. In this paper, we illustrate the concept of arriving at colored honeycombs using two examples of uniform honeycombs, the truncated and the bitruncated cubic honeycombs, shown in Figures 1(a) and (b) respectively. Both of these honeycombs are directly constructed from the only regular honeycomb in three-dimensional Euclidean space – the regular space filling of cubes [4]. Interestingly, the centers of the cells of the bitruncated cubic honeycomb coincide with the body centered cubic (BCC) lattice.

The symmetry group of both the truncated and bitruncated cubic honeycombs is the group [4, 3, 4] generated by four reflections P, Q, R and S satisfying the following relations

$$P^2 = Q^2 = R^2 = S^2 = (PQ)^4 = (QR)^3 = (RS)^4 = (PR)^2 = (PS)^2 = (QS)^2 = e \,.$$

The planes of reflections P and Q, R and S intersect at an angle of $\pi/4$; the planes of reflections Q and R intersect at an angle of $\pi/3$ and the planes of reflections P and R, P and S, Q and S intersect at an angle of $\pi/2$. The planes of the reflections P, Q, R and S are shown in Figures 1(a) and (b), respectively, for the truncated and bitruncated honeycombs.

A method for coloring symmetrical patterns

The following method given in [12], will be applied to arrive at colorings of uniform honeycombs.

Let G denote the symmetry group of the uncolored honeycomb and X the set of cells in the honeycomb. If $C = \{c_1, c_2, \ldots, c_n\}$ is a set of n colors, an onto function $f: X \rightarrow C$ is called an *n-coloring* of X. To each $x \in X$ is assigned a color in C. The coloring determines a partition $\mathcal{P} = \{f^{-1}(c_i): c_i \in C\}$ where $f^{-1}(c_i)$ is the set of elements of X assigned color c_i.

Let H be the subgroup of G which consists of symmetries in G that effect a permutation of the colors in C. Then $h \in H$ if for every $c \in C$, there is a $d \in C$ such that $h(f^{-1}(c)) = f^{-1}(d)$. This defines an action of H on C where we write $hc := d$ if and only if $h(f^{-1}(c)) = f^{-1}(d)$.

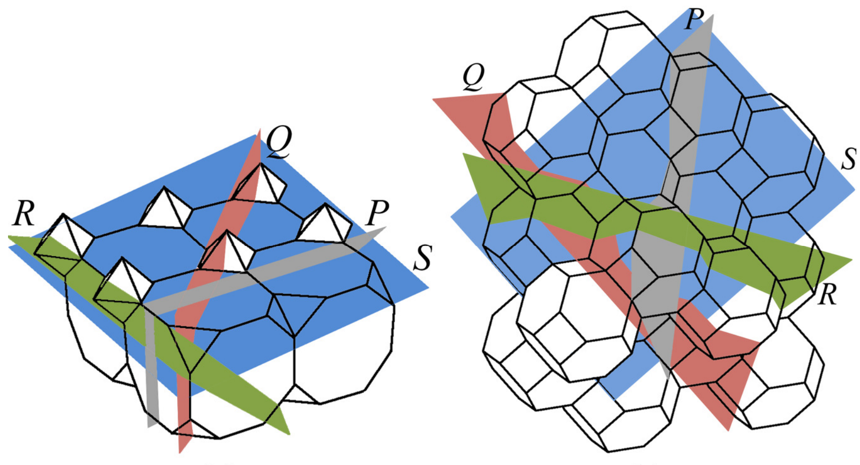


Figure 1: *The* (a) *truncated and* (b) *bitruncated cubic honeycombs with the planes of the reflections P, Q, R and S.*

Since H acts on the set C of colors of X there exists a homomorphism σ from H to Perm(C) where Perm(C) is the group of permutations of C.

Let $c_i \in C$ and denote by O_i the H-orbit of c_i, that is $O_i = Hc_i$. Suppose $J_i = \{h \in H : hc_i = c_i\}$ is the stabilizer of c_i in H. From each H-orbit of X with an element colored c_i, pick one such element. Put these elements together in a set X_i. Then the set of all elements of X that are colored c_i is $J_iX_i = \{jx : j \in J_i, x \in X_i\}$, that is, $f^{-1}(c_i) = J_iX_i$. A one-to-one correspondence results between the sets $O_i = Hc_i$ and $\{hJ_iX_i : h \in G\}$ where hJ_iX_i denotes the image of J_iX_i under h.

As a consequence of the orbit-stabilizer theorem given the assumptions above, we have the following:

Theorem:

1. The action of H on O_i is equivalent to its action on $\{hJ_i : h \in H\}$ by left multiplication.
2. The number of colors in O_i is equal to $[H : J_i]$.
3. The number of H-orbits of colors is at most the number of H-orbits of elements of X.
4. If $x \in X_i$ and $\mathrm{Stab}_H(x) = \{h \in H : hx = x\}$ is the stabilizer of x under the action of H on X then
   (a) $\mathrm{Stab}_H(x) \leq J_i$.
   (b) $|Hx| = [H : J_i] \cdot [J_i : \mathrm{Stab}_H(x)]$

Thus, using the above framework, we outline the steps to obtain a colored uniform honeycomb, where H permutes the colors of the resulting coloring.

1. Pick a cell t from an H-orbit of the elements of X.
2. Determine the finite group S^* of isometries in H which stabilizes t, that is, $S^* = \mathrm{Stab}_H(t)$.
3. Choose a subgroup J of H such that $S^* \leq J$.
4. Apply color c to cell t and to all the cells in the set Jt. If $[H : J] = k$, then Jt is $1/k$ of the cells in the H-orbit where t belongs.
5. Assign a color to every element of the set $\{hJt : h \in H\}$. The set Jt is given color c and each of the remaining $k - 1$ elements of the set gets a different color. In this coloring of the given H-orbit of cells, J will be the stabilizer in H of color c.

To obtain a coloring of a given uniform honeycomb, we consider each H-orbit of cells separately, coloring each orbit with a given set of colors such that H permutes the colors. If two H-orbits of cells are to have a color in common, the subgroup J used should contain the stabilizers of representative tiles from the two H-orbits. Combining the colored orbits of cells will give a colored honeycomb where all elements of H effect a permutation on the set of colors.

Constructing perfect colorings of the truncated and bitruncated cubic honeycombs

In this part of the paper, we discuss the construction of perfect colorings of the truncated and bitruncated cubic honeycombs. Given either a truncated or bitruncated cubic honeycomb, we apply the framework to arrive at colorings where the symmetry group $G = [4, 3, 4]$ of the honeycomb effects a permutation of the colors in the coloring.

In coloring the honeycombs, we will make use of the subgroups of G. (Table 1 gives a list of low index subgroups of G up to conjugacy in G obtained from [8, 16]). In this work, we obtain all perfect colorings where the number of colors used for each G-orbit is at most 4. Any other perfect coloring satisfying the given restriction on the number of colors may be obtained by a symmetry of the honeycomb, a one-to-one change of colors or a combination of both. We consider those colorings where a G-orbit of tiles gets at most 4 colors.

Index	Generators of the subgroup	Index	Generators of the subgroup
2	$A = \langle Q, R, S, PQP\rangle$	4	$F = \langle P, RQ, SRSQ\rangle$
2	$E = \langle P, RQ, S\rangle$	4	$B = \langle Q, R, S, PQRPQP\rangle$
2	$\langle P, RQ, SQ\rangle$	4	$\langle Q, R, PQP, SRS\rangle$
2	$C = \langle P, Q, R, SRS\rangle$	4	$\langle Q, R, PSQP, SRQP\rangle$
2	$\langle Q, R, SP\rangle$	4	$\langle QP, RP, SRSP\rangle$
2	$\langle QP, RP, S\rangle$	4	$\langle RQ, S, PRQP\rangle$
2	$\langle QP, RP, SP\rangle$	4	$\langle RQ, SP, PRQP\rangle$
3	$\langle RPR, RQR, RSR, S, QPQ\rangle$	4	$\langle RQ, SQ, PRQP\rangle$
4	$D = \langle P, Q, R, SRSQRS\rangle$	4	$\langle RQ, PQS\rangle$

Table 1: *The index* 2, 3, 4 *subgroups of* $G = [4, 3, 4]$ *up to conjugacy in* G.

Perfect colorings of the truncated cubic honeycomb. The truncated cubic honeycomb has two G-orbit of cells: the orbit X_1 of octahedra and the orbit X_2 of truncated cubes. The perfect colorings that we will discuss first will involve those colorings where the G-orbits of cells do not share a color; that is, a color that is used in X_1 will not be used in X_2. We will color X_1 first, then X_2.

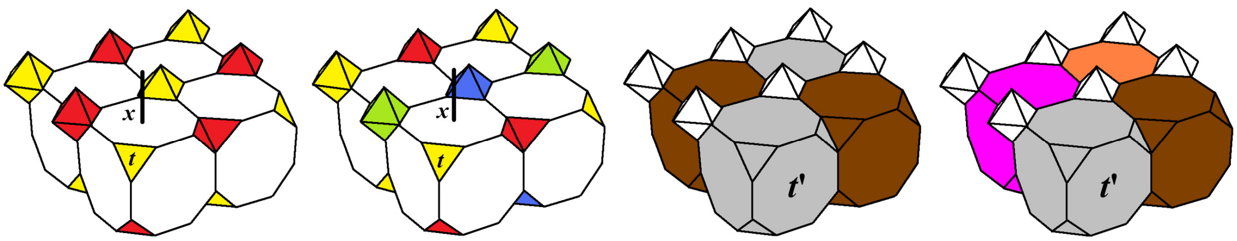
In coloring X_1, we start with the cell labeled t in Figure 2(a). The stabilizer of t in G, $\mathrm{Stab}_G(t)$, is the group generated by Q, R, S, a group of type $\mathbf{O_h}$, also known as the octahedral group. We need to select a subgroup J_1 that satisfies the condition that $\mathrm{Stab}_G(t) \leq J_1$. Using Table 1 we find that the groups $A = \langle Q, R, S, PQP\rangle$ and $B = \langle Q, R, S, PQRPQP\rangle$ are suitable choices for J_1.

To obtain a perfect coloring of X_1 using A, we assign At the color yellow. To color the rest of the orbit, we apply the 2-fold rotation $(PQ)^2$ about x on At to obtain a coloring of two colors shown in Figure 2(a). A perfect coloring of X_1 using $B = \langle Q, R, S, PQRPQP\rangle$ is given in Figure 2(b). The coloring is obtained by assigning all cells in Bt yellow. Then we assign the colors red, blue and green to the other cells by applying the 4-fold rotation PQ about x.

Next, we color the orbit X_2 of truncated cubes. We start with the cell labeled t' in Figure 2(c). The stabilizer of t' in G, $\mathrm{Stab}_G(t')$ is the group $\langle P, Q, R\rangle$ of type $\mathbf{O_h}$. From Table 1, the groups $C = \langle P, Q, R, SRS\rangle$ and $D = \langle P, Q, R, SRSQRS\rangle$ contain $\mathrm{Stab}_G(t')$, thus either C or D may be used to color X_2. Using C and D, we obtain the colorings shown in Figures 2(c) and (d) respectively.

Note that the group G can also be used to color X_1 or X_2 since it contains the stabilizer of every cell. Consequently, all the octahedra or the truncated cubes, respectively, will get one color.

The perfect colorings of the truncated cubic honeycomb where the two G-orbits of cells do not share colors will be obtained by considering the perfect colorings of the octahedra in orbit X_1 and the perfect colorings of the truncated cubes in orbit X_2. Using G, A and B, there are 3 colorings of orbit X_1 and using G, C and D, there are 3 colorings of orbit X_2 that will give rise to 9 perfect colorings of the truncated cubic honeycomb where the orbits X_1, X_2 do not share colors and both X_1 and X_2 get at most 4 colors.



(a) (b) (c) (d)

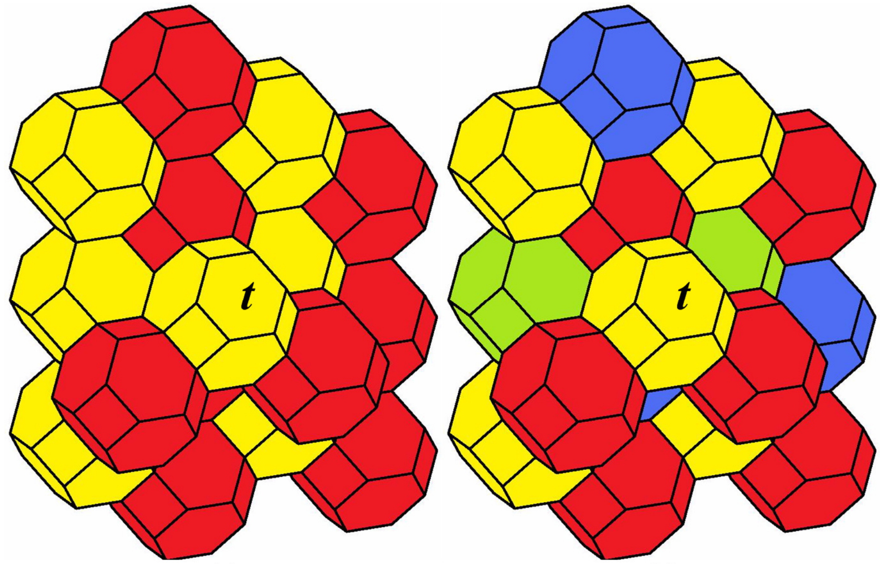
Figure 2: *Perfect colorings of X_1 using*: (a) A and (b) B; *perfect colorings of X_2 using*: (c) C and (d) D.

Perfectly colored honeycombs may also be arrived at by constructing colorings where the G-orbits of cells share colors. If a subgroup J_i is used to color one orbit of cells X_i, it can be used to color another orbit X_j as long as J_i contains the stabilizer of a tile in X_j. Moreover, if a color used to color cell $t \in X_i$ is used to color cells in X_j then the tile $t' \in X_j$ that will be assigned the same color as tile t should have a stabilizer contained in J_i.

We wish to remark that in constructing non-trivial perfect colorings of the truncated cubic honeycomb, the G-orbits of cells X_1 and X_2 cannot share colors. The subgroups A and B, for example cannot be used to color cells in X_2 since these groups do not contain a stabilizer of a cell in X_2. Similarly, the subgroups C and D cannot be used to color cells in X_1 since these groups do not contain a stabilizer of a cell in X_1.

Perfect colorings of the bitruncated cubic honeycomb. To color the bitruncated cubic honeycomb, we first note that this honeycomb has only one type of cell – the truncated octahedron. The symmetry group G of the bitruncated cubic honeycomb is cell-transitive. This means that we only have one G-orbit of cells to color. Since the symmetry group of the uncolored bitruncated cubic honeycomb is also $G = [4, 3, 4]$, we will use the list provided in Table 1 to choose the subgroups that we can use to color.

First, consider the truncated octahedron labeled t in Figure 3(a). The stabilizer of t in G is $\langle P, Q, R\rangle$ a group of type $\mathbf{O_h}$. Aside from G, the groups $C = \langle P, Q, R, SRS\rangle$, $D = \langle P, Q, R, SRSQRS\rangle$ contain $\langle P, Q, R\rangle$ and may be used to arrive at perfect colorings of the bitruncated cubic honeycomb. The colorings of the entire honeycomb using C and D are given in Figures 3(a) and (b), respectively.



(a) (b)

Figure 3: *Perfect colorings of the bitruncated cubic honeycomb using* (a) *C and* (b) *D.*

Semi-perfect colorings of the bitruncated cubic honeycomb

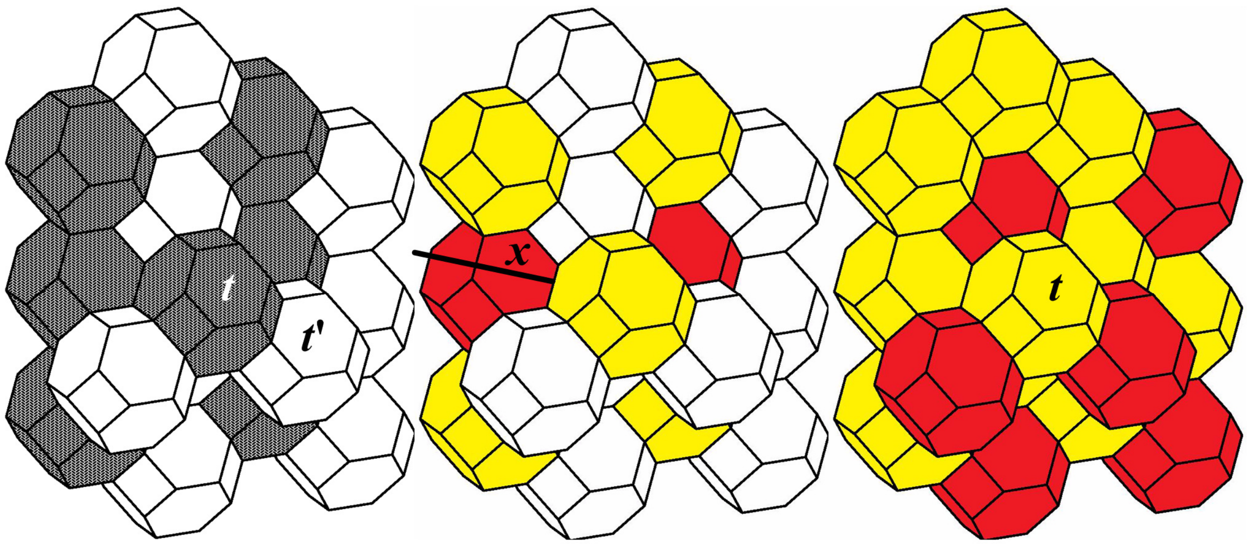
In this part of our work, we illustrate the construction of semi-perfect colorings of the bitruncated cubic honeycomb. In this case, the group H consisting of elements that permute the colors in a given coloring is an index 2 subgroup of G. Following the framework presented earlier, the first step is to choose an index 2 subgroup H and determine the H-orbits of cells. Then we proceed by coloring each H-orbit of cells separately.

For our first example, let us construct a semi-perfect coloring where in particular the subgroup $C = \langle P, Q, R, SRS \rangle$, permutes the colors in the given coloring. There are two C-orbits of cells. As shown in Figure 4(a), the set of “patched” cells is the C-orbit X_1, while the set of white cells is the C-orbit X_2.

To color X_1, we first choose our starting cell t labeled in Figure 4(a). Note that $\mathrm{Stab}_C(t) = \langle P, Q, R \rangle$ of type $\mathbf{O_h}$. The subgroup $D = \langle P, Q, R, SRSQRS \rangle$ of C contains $\langle P, Q, R \rangle$, so that we let $J_1 = D$. We assign the color yellow to the set Dt and red to the set aDt, to obtain the coloring shown in Figure 4(b) (a is the two-fold rotation with axis x labeled in Figure 4(b)).

To color X_2, note that if $t' \in X_2$ then $\mathrm{Stab}_C(t') \leq C$, so we can use C to color X_2 and X_2 gets a single color. Assuming the color white is used to color X_2, the semi-perfect coloring we obtain is the coloring given in Figure 4(b).

As a second example, let us construct a semi-perfect coloring where $E = \langle P, RQ, S \rangle$ is the group consisting of elements that will permute the colors of the coloring. In this case, all the truncated octahedra in the honeycomb will form one orbit of cells under E, so we only have one E-orbit to color. Consider the cell labeled t in Figure 4(c) where $\mathrm{Stab}_E(t) = \langle P, RQ \rangle$. The subgroup $F = \langle P, RQ, SRSQ \rangle$ of E has $\langle P, RQ \rangle$ as a subgroup, so that we may use F to color the honeycomb semi-perfectly. The resulting semi-perfect coloring is shown in Figure 4(c).



(a) (b) (c)

Figure 4: (a) *Two C-orbit of tiles of the bitruncated cubic honeycomb* (*C* = < *P*, *Q*, *R*, *SRS*>);
(b)-(c) *Semi-perfect colorings of the bitruncated cubic honeycomb.*

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