Concave Hexagons

Paul Gailiunas
25 Hedley Terrace, Gosforth
Newcastle, NE3 1DP, England
email: p-gailiunas@argonet.co.uk

Abstract

The tilings (n.3.n.3) exist in the spherical, Euclidean or hyperbolic plane, depending on whether n is less than, equal to, or greater than 6. In all cases the dual tiling consists of rhombi, which can be taken in pairs to form "regular" concave hexagons. In the case of the spherical examples the tilings can be illustrated by colouring the faces of rhombic polyhedra. In the Euclidean plane "regular" concave hexagons allow tilings that cannot be constructed from the dual (6.3.6.3) tiling, some of which allow analogous tilings of non-"regular" concave hexagons. Some Escher-like designs are derived from such tilings.

Some of the possibilities in the hyperbolic plane are briefly considered.

Some Concave Spherical Hexagons

Previously [1] I described polyhedra that can be produced by removing parts of the small stellated dodecahedron, having fifteen faces that are equilateral triangles, and fifteen that are concave pentagons (figure 1). The two types of face are always paired, and it is convenient to think of a pair as some kind of irregular folded hexagon, which can be projected onto the surface of a sphere with centre coincident with the centre of the original small stellated dodecahedron.

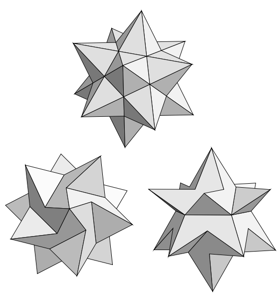

Figure 1: *The small stellated dodecahedron, and two views of a polyhedron derived from it that has five-fold symmetry.*

These polyhedra have two types of vertex, twelve that are vertices of the convex hull (a regular icosahedron), and twenty that are vertices of the regular dodecahedron from which the original stellation is derived. It follows that in the projection they are the vertices of a spherical triacontahedron, and the spherical hexagons consist of pairs of spherical rhombi. The resulting spherical tilings can be conveniently indicated by colouring faces of the triacontahedron in pairs (figure 2).

Figure 2: *Two polyhedra with triacontahedra coloured in corresponding ways. One pair has 5-fold symmetry, the other has 3-fold symmetry.*

Since there are fifteen hexagons the only possible symmetrical polyhedra have 3-fold or 5-fold symmetry, as illustrated in figure 2. In both cases the hexagons around a symmetrical vertex can be arranged in either sense, so each polyhedron exists in four forms (two enantiomorphic pairs). Mirror symmetry is not possible since any plane would have to bisect at least one hexagon, because fifteen is an odd number, and it would cut adjoining hexagons non-symmetrically. There is a large number of non-symmetrical arrangements.

The triacontahedron is the dual of the icosidodecahedron, with vertex symbol (5.3.5.3). There are two other face-regular rhombic polyhedra, the rhombic dodecahedron, the dual of the cuboctahedron, (4.3.4.3), and the cube, the dual of the octahedron, (3.3.3.3). In both of these cases the rhombic faces can be taken in pairs corresponding to tilings of concave hexagons on the sphere.

In the case of the rhombic dodecahedron there are six concave hexagons so that, although it is possible to place four hexagons symmetrically around a 4-valent vertex, this leaves only two hexagons, so that complete 4-fold symmetry is impossible. There is a natural arrangement with 3-fold symmetry that corresponds to Coffin's "pennyhedron puzzle" [2].

It is easy to make three pairs of adjacent cubic faces, in only one way, with 3-fold symmetry.

Euclidean Tilings

Since the Euclidean plane is infinite there is no limit to the number of ways edges can be removed from the dual (6.3.6.3) tiling to produce concave hexagons, even if we consider only periodic tilings with rotational symmetry. It is instructive, however, to consider analogues of the symmetrical polyhedra in figure 2. A general strategy is to start at a symmetrical vertex and to add successive rings of hexagonal tiles, noting that the 3-valent vertices occur at the concave vertex and at the two off-centre 120• angles. At the local level the main difference is the addition of extra tiles at the 5-fold vertices to make them 6-fold.

In the lower half of the right-hand illustration in figure 1 the hexagons (concave pentagon + triangle) in the second (equatorial) ring have their two concave edges meeting edges of one hexagon in the first (lower) ring. The third (upper) ring is like the first ring: five hexagons around a symmetrical vertex. The Euclidean analogue (figure 3) consists of six concave hexagonal tiles combined to form a regular hexagon (the first ring) with tiles at its corners (the second ring). Regular hexagons corresponding to the third ring meet the tiles of the second ring in the same way as those in the spherical case, with convex 3-valent vertices.

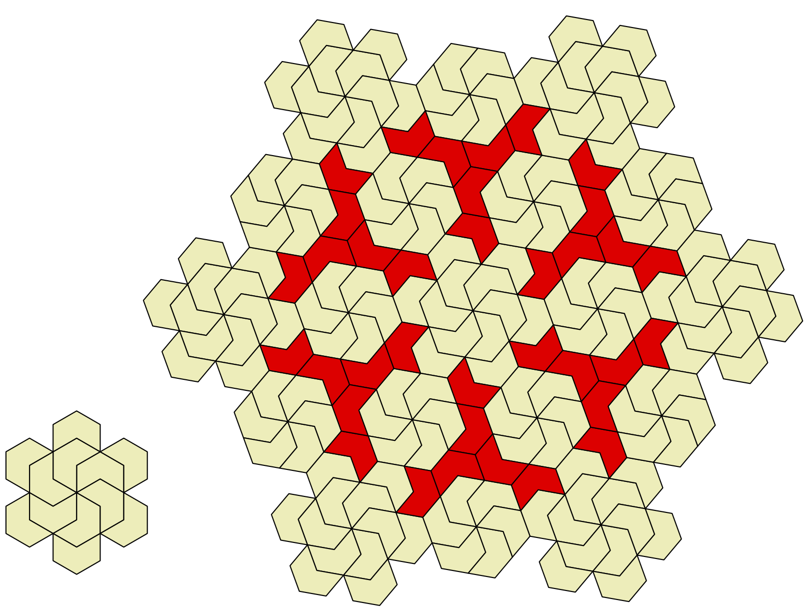

Figure 3: *A Euclidean analogue of the spherical tiling with 5-fold symmetry.*

There is only one hexagonal plane symmetry group without mirrors, p6, or **632** in orbifold notation. The patches corresponding to the first two rings are centred on 6-fold axes, but the patches corresponding to the third ring, which also have local 6-fold symmetry, are centred on 2-fold axes of the tiling. The patches formed by the tiles that must be added (red in the figure) are centred on the 3-fold axes of the tiling.

Applying the same process from the top of the right-hand illustration of figure 1, so the order of rings is reversed, generates a different tiling pattern (figure 4).

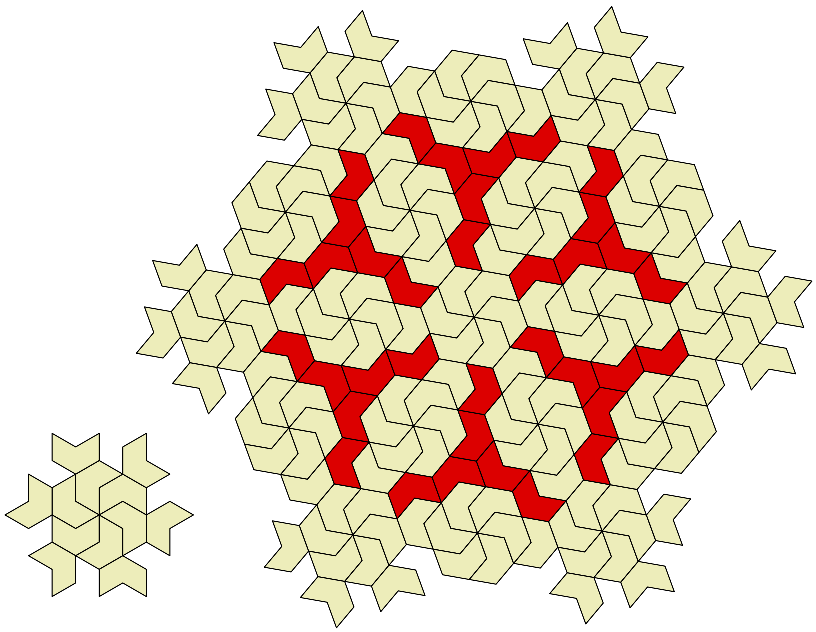

Figure 4: *An alternative Euclidean analogue of the spherical tiling with 5-fold symmetry.*

Spiral Tilings

All of the tilings considered so far are derived from duals of (n.3.n.3), and no alternative is possible on the sphere. In the Euclidean plane, however, the angles of the concave hexagons are commensurable, so it is possible to disregard the distinction between 3-valent and 6-valent vertices. In particular chevron arrangements are possible, which allows the two spiral tilings described towards the end of [3] (figure 5).

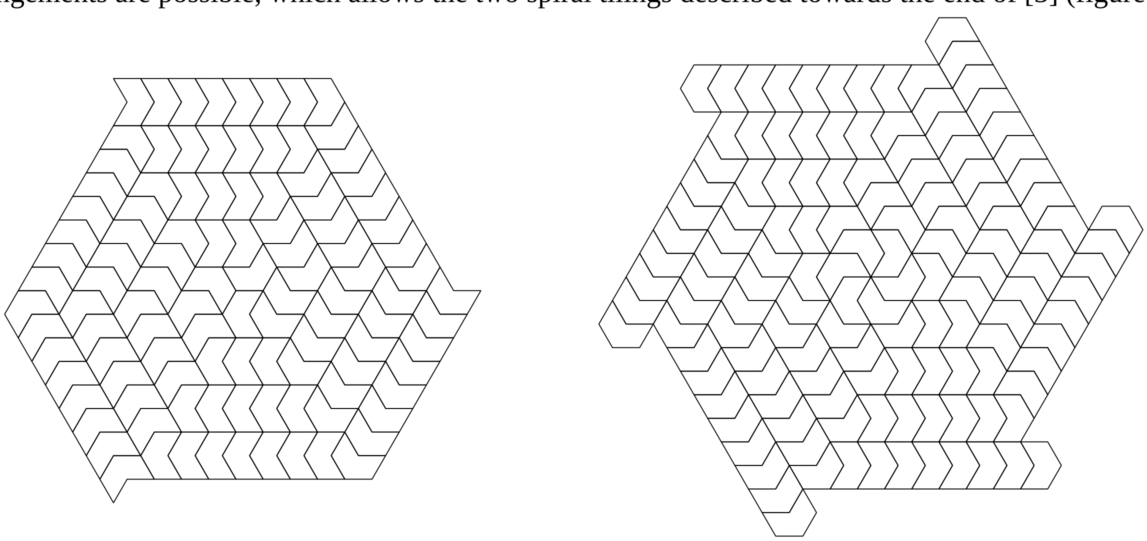

Figure 5: *The two spiral tilings possible with 60• concave hexagons.*

The patches of tiles around the central vertex, six in the 3-armed example, eighteen in the 6-armed example, observe the vertex restrictions, and can be seen as Euclidean analogues of some symmetrical patches that occur in spherical tilings. The straight sections of the spiral paths, however, do not fit on the dual (6.3.6.3) tiling, and it is not possible to identify vertices as either 3- or 6-valent. The pairs of tiles at the corners of the spiral paths, where there is a 60• turn, are the only neighbours that preserve vertex type in this way.

Edge-deformation

The tiling patterns of M.C.Escher are all periodic, or derived from periodic drawings [4] (obviously apart from those that involve no element of symmetry at all and the spherical examples), and the standard methods of construction rely on periodicity to ensure that complicated shapes will fit together properly. Inspection of the spiral tilings in [1] suggests that the usual techniques of edge-deformation will not work with them, since edges meet copies of many other edges, which in turn meet copies of many edges, so that there is no consistent way to deform the edges without generating a large number of differently shaped tiles. There are some circumstances. however, in which spiral tilings can form the basis of Escher-like patterns with only two shapes of tile.

If an edge is a straight line segment it has two lines of (local) symmetry, itself and its perpendicular bisector (and, of course, together they generate a rotational symmetry of order 2). The symmetry can be broken by losing either line of symmetry, or both. Breaking the symmetry along the line of an edge, for example by the edge becoming a concave or a convex circular arc, is considered in [5]. Figure 6 shows a modification of the 3-armed spiral in figure 5 with edges that are all circular arcs.

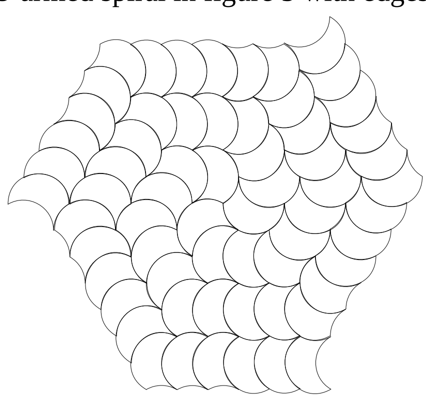

Figure 6: *A modified version of the 3-armed spiral in figure 5, having edges that are circular arcs.*

Such local symmetry-breaking of the edges of a tile derived from a regular polygon is possible only when the tiling does not rely on special circumstances, for example the tile having only one concave vertex (see [5] for details). The 6-armed spiral in figure 5, and the periodic tilings in figures 3 and 4 depend on particular properties of hexagons, so there is no equivalent edge-deformation that preserves the monohedral tilings.

If edges are replaced by a line of arbitrary shape we need to consider its orientation between vertices as well as its orientation between faces (concave/convex in figure 6, for example). The local symmetry of the edges in a perpendicular line could, in principle, be broken in tilings such as those in figures 3 and 4, where there is a consistency in the way vertices are arranged. For example edges could be labelled with arrows directed from 3-valent vertices to 6-valent vertices.

Figure 7 shows the 3-armed spiral of figure 5 with the edges replaced by arbitrary curves. Two shapes of tile are needed because there is no way to direct the edges between vertices in a consistent way. Notice that the shapes of tile alternate along the spiral path, except at the corners, where there are pairs of the same shape, as expected from the earlier observation about vertex type in spiral tilings. In fact spiral tilings of this type exist for all orders of rotational symmetry, since the "regular" concave hexagons that consist of two 60• rhombi can be transformed into non-"regular" concave hexagons that consist of two different rhombi, that will still tile an infinite wedge that can form the fundamental domain of some rotational symmetry. Figure 7 also shows a 4-armed spiral, previously described in [1], with edges substituted by the same arbitrary curve. Tiles of the same shape form v-shaped zones, and the structure is closely related to the way this 4-armed spiral is coloured in [6].

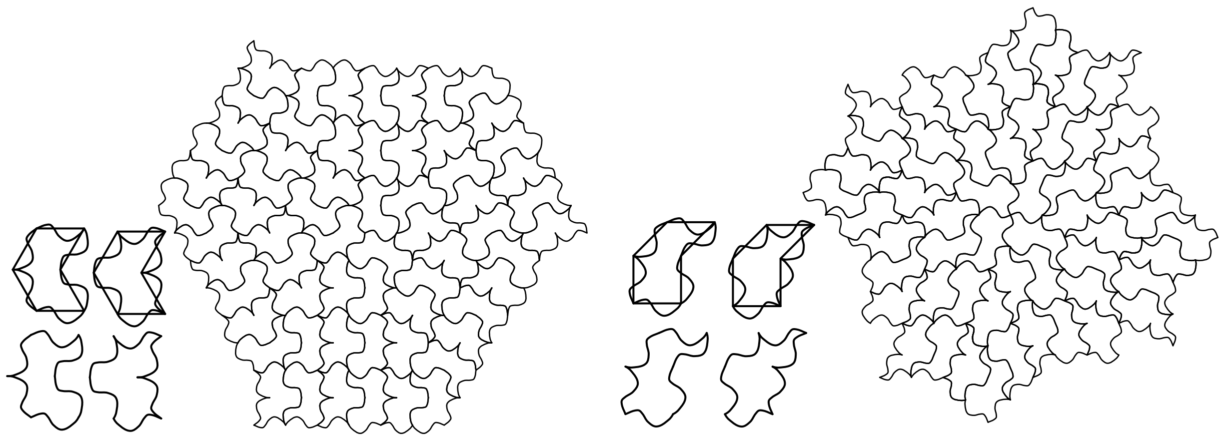

Figure 7: *A modified version of the 3-armed spiral in figure 5, and an analogous 4-armed spiral.*

While one of the tile shapes, especially in the 3-armed case, could make quite a convincing bird, the other does not suggest any realistic form, however stylized. In the case of the 60• concave hexagon it is possible to arrange the curved edges so that they start and finish with straight sections that are coincident in some parts of the tile boundary (figure 8).

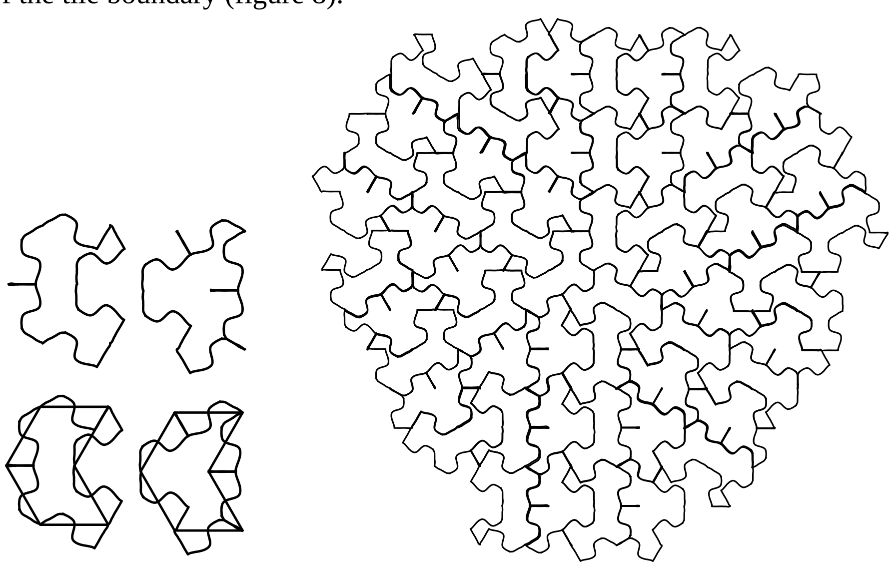

Figure 8: *An alternative version of the 3-armed spiral.*

This change converts the "awkward" tile into quite a reasonable bird, and the bird tile into a more-or-less dog shape (figure 9). The problem with these substitutions is that the same shape must be substituted for every edge (although in different orientations) so that there needs to be a compromise: what produces a better shape at one place makes a worse one at another. The dogs in figure 9, for example, do not bear close scrutiny, although the birds (doves maybe?) are reasonably convincing.

In a tiling such as that of figure 6 the concave edge of one tile coincides with the convex edge of its neighbour. This is the same as the relationship between directed edges if they are considered in an intrinsic way, i.e. clockwise or anticlockwise around the tile. A directed edge in a tiling is clockwise around one of its tiles and anticlockwise around the other. This observation suggests another kind of edge-deformation, where the orientation of an edge is either clockwise and concave, or anticlockwise and convex. A consequence is that all the edges in the tiling are related by direct isometries, and there are no mirror images.

Figure 11 shows an example of this type of edge-deformation applied to the tilings in figure 5. The edges are directed in the same way as those in figure 7 (alternately), and this determines their orientation with respect to faces ("concave" or "convex"). Again there are two shapes of tile, for the same reason as before. Since the central patch of the 6-armed spiral tiling can be derived from the dual (6.3.6.3) tiling, and the rest of the tiling consists of chevron elements, they are consistent with it.

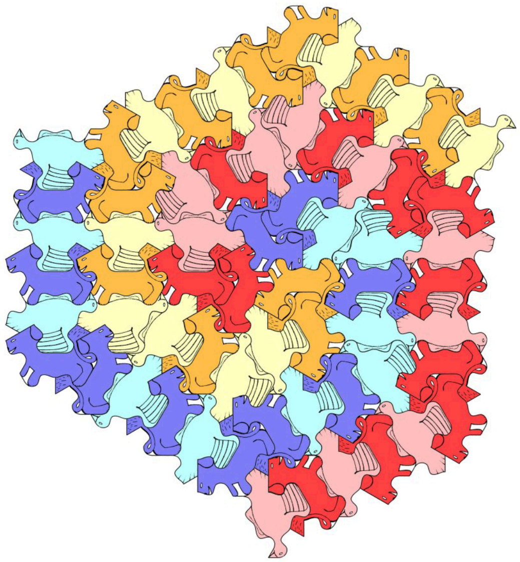

Figure 9: *Figure 8 with designs added to the tiles, and coloured to show the spirals.*

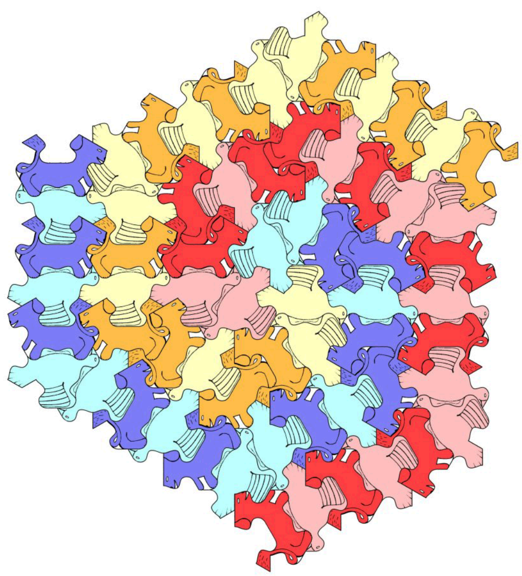

Figure 10: *Interchanging tile shapes still gives a consistent tiling.*

In general spiral tilings of the type described in [1] require mirror-image copies of tiles, but if all the edges are deformed so that they are related by direct isometries, mirror image copies will not fit. In this case four different shapes of tile are needed: a pair where corresponding edges are rotated by 180° with respect to each other, and a pair based on the mirror image but with edges that are direct copies from the first pair, again where corresponding edges are rotated by 180° with respect to each other. The 3-armed spiral in figure 11 is a special case, because it is based on "regular" concave hexagons that have mirror symmetry, so the two pairs are the same.

The same idea can be apppied to the tiling illustrated in figure 6. Edges that are some random shape are directed according to whether the the circular arc is concave or convex. This time in general two shapes of tile are sufficient, and figure 12 is equivalent to figure 7, but with tiles related by direct isometries only.

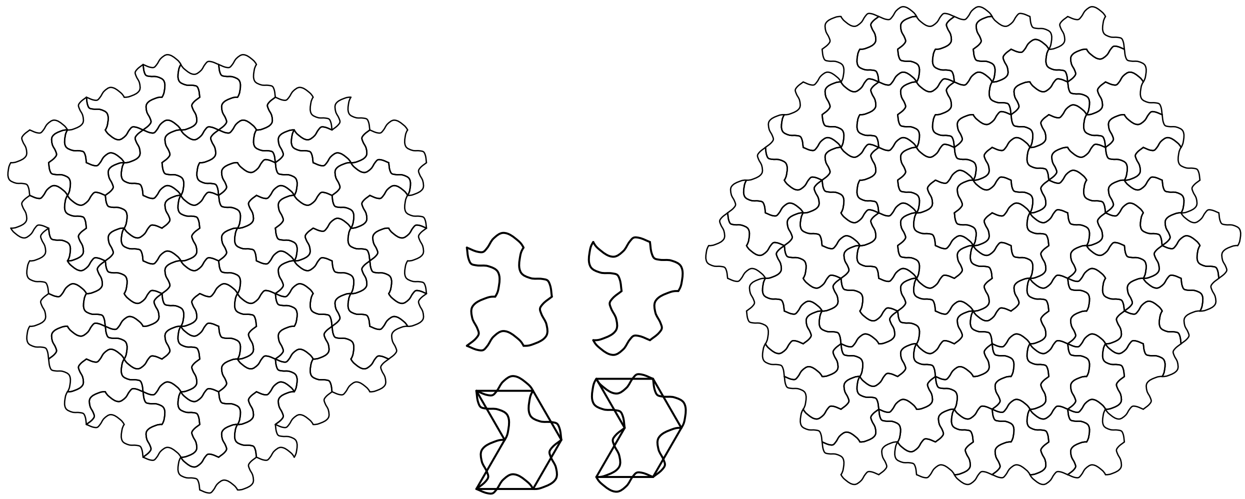

Figure 11: *Modified versions of the spiral tilings in figure 5.*

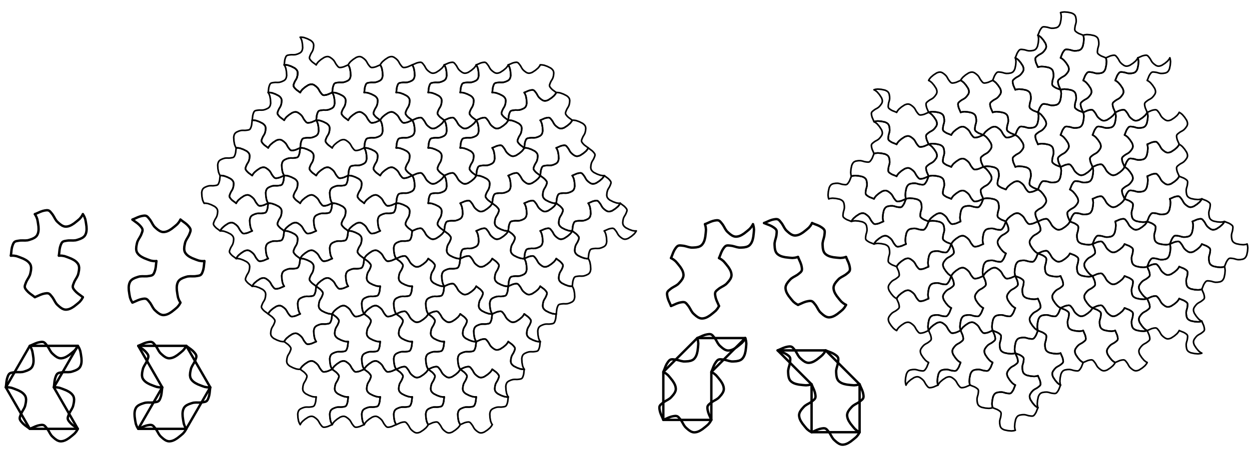

Figure 12: *A different edge-deformation of the tilings in figure 7.*

Hyperbolic Tilings

The tilings (n.3.n.3) lie in the hyperbolic plane if n>6, and once again tilings of concave hexagons can be produced by removing edges from the dual tilings. As with spherical geometry the size of polygons depends on the sum of their internal angles, and a problem with images of these tilings, which is worse for higher values of n, is that most of the interesting structure gets pushed to the edges, whichever projection is used.

When n=7, the image is not too distorted, and Douglas Dunham has presented a tiling of butterflies that indicates how the tiles might look, although he uses edge-deformations that are less restricted than the ones needed for spiral tilings [7]. In this case the angles of the concave hexagons are incommensurable, so that there is a distinction between 3-valent and 7-valent vertices in any tiling. Commensurable angles occur whenever n is divisible by 3, so the smallest value in the hyperbolic plane is 9. Analogues of the Euclidean 3-armed spiral should be possible, although extra tiles must be fitted into the equivalent of the rows of chevrons. When n=12 two tiles will fit into the concave angle, and it seems that some kind of branching spiral tiling might be possible. Even when n=9 projections of hyperbolic tilings are pushed quite far to the perimeter, so it is likely that images of these tilings will be quite disappointing.

Not a lot of software exists for investigating hyperbolic tilings, and what is available is designed around full symmetry, typically isohedral tilings. The spiral type of tilings have a single centre of symmetry, and they are not easily investigated. The best strategy seems to involve colouring tilings of triangles manually. Very little has been attempted, and there is much to be discovered, but the results are unlikely to lead to striking images for the reasons already discussed.

Conclusion and Further Possibilities

Tilings of concave hexagons exist in spherical, Euclidean and hyperbolic geometries, and they provide a range of decorative possibilities. The technique of edge-deformation has been used in all three geometries by Escher [4], and it can be applied in these examples. Grünbaum and Shephard's classification of isohedral tilings [8] can be used to find possibilities in the Euclidean plane that can be produced by deleting edges from IH30-37.

There are more constraints on edge-deformations to ensure they will work with spiral tilings of concave hexagons. In particular every edge must be the same shape. Some possibilities have been described, and they illustrate aspects of structure that are not apparent in the examples previously described in [3] that have edges that are straight lines. There are related tilings described in [5], with more illustrated in the extended online version [9], to which the technique of edge-deformation could be applied, providing further decorative possibilities, and maybe additional insight into their structures.

All of the tiles considered are compounds of two rhombi, so they have special properties, being zonogons. There are other classical tilings and polyhedra that have quadrilateral faces that are not rhombi. It is unlikely that they will provide such a rich source of possibilities, but some other tilings of concave hexagons can be generated by deleting edges from them, with properties yet to be investigated.

References

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