APPENDIX
Combinatorica Poetica
Counting and Visualizing Rhyme Patterns in Sonnets

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■ Introduction

This notebook contains computations and proofs for some of the properties of the generating functions $G_k(x)$, for 1≤k, whose coefficients appear to be the numbers in the columns of Table 2 that I present in the paper "Combinatoria Poetica: Counting and Visualizing Rhyme Patterns in Sonnets". The generating functions are presented in two ways: (i) as products of linear hyperbolic functions, and (ii) as power series with coefficients that can be expressed in a closed form.

In the first section I show calculations for the counts in Table 2 of the paper using the formula for the coefficients of the generating functions. In the second section I implement the generating functions in *Mathematica* as products and compute the Taylor expansions for the first seven generating functions using the *Mathematica* function Series[ ]. The counts for poems of up to 14 lines with even end rhyme patterns are covered by this computation. In the third section I establish a recurrence relation for the generating functions and conclude by establishing the same recurrence for the coefficents of the associated power series.

■ The Generating Functions $G_k(x)$

(1) The generating function $G_k(x)$ for the k^{th}column $\{c_{i+k,k}\}_{i\geqslant 0}$ in the Even Rhymes Triangle is

$$G_k(x) = \sum_{i=0}^{\infty} c_{i+k,k}\, x^i = \prod_{j=1}^{k} \frac{2j-1}{1-j^2 x}\,.$$

(2) The closed form expression for the i^{th} coefficient $c_{i+k,k}$ of the k^{th} generating function $G_k(x)$ is

$$c_{i+k,k} = \frac{1}{2^{k-1}\,k!} \times \sum_{j=1}^{k} \left((-1)^{k+j} \binom{2k}{k+j} \left(j^2\right)^{k+i}\right)$$

therefore, the k^{th} generating function $G_k(x)$ can be written as

$$G_k(x) = \frac{1}{2^{k-1}\,k!} \sum_{i=0}^{\infty} \left(\sum_{j=1}^{k} (-1)^{k+j} \binom{2k}{k+j} \left(j^2\right)^{k+i} \right) x^i$$

■ Section 1: The Counting Function evenRhymesNumbers[n , k]

The function evenRhymesNumbers[ ] implements the closed form expression for the coefficients of the generating functions $G_k(x)$, 1≤k:

$$\frac{1}{2^{k-1}\,k!} \sum_{j=1}^{k} (-1)^{k+j} \binom{2k}{k+j} \left(j^2\right)^{k+i}$$

In the function implementation the term i+k is replaced by the variable n. Note that Factorial2[2k] is equivalent to 2^k k! .

```
Clear[evenRhymesNumbers]
evenRhymesNumbers[n_, k_] := Which[n < k, 0, n == 1, 1, True,
  2 / Factorial2[2 k] Sum[(-1) ^ (k + j) Binomial[2 k, k + j] j ^ (2 n), {j, 1, k}]]
```

```
Clear[evenRhymesTable]
evenRhymesTable[n_, k_] := Table[evenRhymesNumbers[a, b], {a, 1, n}, {b, 1, k}]
```

```
TableForm[evenRhymesTable[7, 7]]
```

1	0	0	0	0	0	0
1	3	0	0	0	0	0
1	15	15	0	0	0	0
1	63	210	105	0	0	0
1	255	2205	3150	945	0	0
1	1023	21 120	65 835	51 975	10 395	0
1	4095	195 195	1 201 200	1 891 890	945 945	135 135

```
Clear[evenRhymesTriangle]
evenRhymesTriangle[n_] := Table[evenRhymesNumbers[a, b], {a, 1, n}, {b, 1, a}]
```

```
TableForm[evenRhymesTriangle[7]]
```

1						
1	3					
1	15	15				
1	63	210	105			
1	255	2205	3150	945		
1	1023	21 120	65 835	51 975	10 395	
1	4095	195 195	1 201 200	1 891 890	945 945	135 135

Section 2: The Coefficients of the Power Series $G_k(x)$

The function g[ ] defines the generating functions $G_k(x)$ as products of hyperbolic functions; the first seven instances are displayed.

```
Clear[g]
g[k_, x_] := Product[(2 j - 1) / (1 - j^2 x), {j, 1, k}]
```

```
TableForm[Table[g[k, x], {k, 1, 7}]]
```

$$\frac{1}{1-x}$$

$$\frac{3}{(1-4x)(1-x)}$$

$$\frac{15}{(1-9x)(1-4x)(1-x)}$$

$$\frac{105}{(1-16x)(1-9x)(1-4x)(1-x)}$$

$$\frac{945}{(1-25x)(1-16x)(1-9x)(1-4x)(1-x)}$$

$$\frac{10\,395}{(1-36x)(1-25x)(1-16x)(1-9x)(1-4x)(1-x)}$$

$$\frac{135\,135}{(1-49x)(1-36x)(1-25x)(1-16x)(1-9x)(1-4x)(1-x)}$$

The corresponding power series expansion columnPoly[ ] using the two functions Normal[ ] and Series[ ] computes the Taylor series expansion of g[ ] about x=0 up to degree d; the first seven instances of polynomials of degree five are displayed.

```
columnPoly[k_, x_, d_] := Normal[Series[g[k, x], {x, 0, d}]]
```

```
TableForm[Table[columnPoly[k, x, 5], {k, 1, 7}]]
```

$1 + x + x^2 + x^3 + x^4 + x^5$
$3 + 15\,x + 63\,x^2 + 255\,x^3 + 1023\,x^4 + 4095\,x^5$
$15 + 210\,x + 2205\,x^2 + 21\,120\,x^3 + 195\,195\,x^4 + 1\,777\,230\,x^5$
$105 + 3150\,x + 65\,835\,x^2 + 1\,201\,200\,x^3 + 20\,585\,565\,x^4 + 341\,809\,650\,x^5$
$945 + 51\,975\,x + 1\,891\,890\,x^2 + 58\,108\,050\,x^3 + 1\,637\,971\,335\,x^4 + 44\,025\,570\,225\,x^5$
$10\,395 + 945\,945\,x + 54\,864\,810\,x^2 + 2\,614\,321\,710\,x^3 + 112\,133\,266\,245\,x^4 + 4\,521\,078\,857\,295\,x^5$
$135\,135 + 18\,918\,900\,x + 1\,640\,268\,630\,x^2 + 114\,359\,345\,100\,x^3 + 7\,061\,340\,371\,085\,x^4 + 404\,779\,703\,328\,000\,x^5$

Section 3: Properties of the Generating Functions $G_k(x)$, 1 ≤ k

A Recurrence for the Product Formula for the Generating Functions $G_k(x)$, 1 ≤ k

CLAIM:
The recurrence for the coefficients $c_{i+k,k}$ for the power series functions:
$c_{i+1,1} = 1$ for all $0 \leqslant$ i,
$c_{i+k,k} = 0$ for all i < 0 and $1 \leqslant$ k, and
$c_{i+k,k} = (2\text{ k} - 1) \times c_{i+k-1,k-1} + k^2 \times c_{i+k-1,k}$ for all $0 \leqslant$ i and $1 \leqslant$ k,
implies the product expression
$G_k(x) = \sum_{i=0}^{\infty} c_{i+k,k}\, x^i = \prod_{j=1}^{k} \frac{2\,j-1}{1-j^2\,x}$
of hyperbolic functions for the generating functions.

PROOF:
Note that as a consequence of the recurrence when i=0, the equalities $c_{1,1}$= 1 and $c_{k,k}$ = (2k - 1) $c_{k-1,k-1}$, for all k > 1, holds.

BASIS OF INDUCTION: k = 1 :
$G_1(x) = \sum_{i=0}^{\infty} c_{i+1,1}\, x^i = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$.

INDUCTION STEP: k > 1 :
Starting with the power series expansion for $G_k(x)$:
$G_k(x) = \sum_{i=0}^{\infty} c_{i+k,k}\, x^i = \sum_{i=1}^{\infty} \left((2\,k-1)\; c_{i+k-1,k-1} + k^2\, c_{(i-1)+k,k}\right) x^i + c_{k,k}\, x^0$
$= (2\text{k} - 1) \sum_{i=1}^{\infty} c_{i+k-1,k-1}\, x^i + k^2 x \sum_{i=1}^{\infty} c_{(i-1)+k,k}\, x^{i-1} + (2\text{k} - 1)\, c_{k-1,k-1}\, x^0$
$= (2\text{k} - 1) \sum_{i=0}^{\infty} c_{i+k-1,k-1}\, x^i + k^2 x \sum_{i=0}^{\infty} c_{i+k,k}\, x^i$
$= (2\text{k} - 1)\, G_{k-1}(x) + k^2 x\, G_k(x)$,
therefore, from $(1 - k^2 x)\, G_k(x) = (2\text{k} - 1)\, G_{k-1}(x)$ or $G_k(x) = \frac{2\,k-1}{1-k^2\,x} G_{k-1}(x)$
the product formula follows.

The Recursion Relation for the Closed Form of the Coefficients of $G_k(x)$, 1 ≤ k

CLAIM:
The coefficients $c_{i+k,k} = \frac{1}{2^{k-1}\,k!} \times \sum_{j=1}^{k} \left((-1)^{k+j} \binom{2\,k}{k+j} \left(j^2\right)^{k+i}\right)$ for the power series functions satisfy the recurrence:
$c_{i+1,1} = 1$ for all $0 \leqslant$ i,
$c_{i+k,k} = (2\text{ k} - 1) \times c_{i+k-1,k-1} + k^2 \times c_{i+k-1,k}$ for all $0 \leqslant$ i and $1 \leqslant$ k.

PROOF:
The claim is proved by induction on k using the closed form expression for the i^{th} coefficient $c_{i+k,k}$:

$c_{i+k,k} = \frac{1}{2^{k-1}\,k!} \times \sum_{j=1}^{k} \left((-1)^{k+j} \binom{2k}{k+j} \left(j^2\right)^{k+i} \right)$, for all $0 \leqslant i \in \mathbb{N}$ and all $1 \leqslant k \in \mathbb{N}$.

BASIS OF INDUCTION: k = 1 and all $0 \leqslant i$:

$$c_{i+1,1} = \frac{1}{1\times 1!} \times \sum_{j=1}^{1} \left((-1)^{1+j} \binom{2\times 1}{1+j} \left(j^2\right)^{1+i} \right) = \frac{1}{1} (-1)^{1+1} \binom{2}{1+1} \left(1^2\right)^{1+i} = 1^{1+i} = 1$$

INDUCTION STEP: k > 1 and all $0 \leqslant i$:

Starting with the right hand side of the recurrence and simplifying the expression establishes equality with the left hand side.

$$(2k-1) \times c_{i+k-1,k-1} + k^2 \times c_{i+k-1,k}$$

$$= (2k-1) \times \frac{1}{2^{k-2}\,(k-1)!} \times \sum_{j=1}^{k-1} \left((-1)^{(k-1)+j} \binom{2(k-1)}{(k-1)+j} \left(j^2\right)^{(k-1)+i} \right) + k^2 \times \frac{1}{2^{k-1}\,k!} \times \sum_{j=1}^{k} \left((-1)^{k+j} \binom{2k}{k+j} \left(j^2\right)^{k+(i-1)} \right)$$

$$= \frac{1}{2^{k-1}\,k!} \times \sum_{j=1}^{k-1} \left((2k-1)(2k)(-1)^{(k-1)+j} \frac{(2(k-1))!}{(2(k-1)-(k-1)-j)!\,((k-1)+j)!} + k^2 (-1)^{k+j} \frac{(2k)!}{(2k-k-j)!\,(k+j)!} \right) \left(j^2\right)^{k+i-1} + k^2 \times \frac{1}{2^{k-1}\,k!} \times (-1)^{2k} \times \binom{2k}{2k} \times \left(k^2\right)^{k+i-1}$$

$$= \frac{1}{2^{k-1}\,k!} \times \sum_{j=1}^{k-1} \left((-1)^{(k-1)+j} \frac{(2k)!}{(k-j-1)!\,(k+j-1)!} + k^2 (-1)^{k+j} \frac{(2k)!}{(k-j)!\,(k+j)!} \right) \left(j^2\right)^{k+i-1} + \frac{1}{2^{k-1}\,k!} \times (-1)^{k+k} \times \binom{2k}{2k} \times \left(k^2\right)^{k+i}$$

$$= \frac{1}{2^{k-1}\,k!} \times \sum_{j=1}^{k-1} (-1)^{k+j} \frac{-(2k)!\,(k-j)(k+j) + k^2 (2k)!}{(k-j)!\,(k+j)!} \left(j^2\right)^{k+i-1} + \frac{1}{2^{k-1}\,k!} \times (-1)^{k+k} \times \binom{2k}{2k} \times \left(k^2\right)^{k+i}$$

$$= \frac{1}{2^{k-1}\,k!} \times \sum_{j=1}^{k-1} (-1)^{k+j} \frac{(2k)!\,j^2}{(k-j)!\,(k+j)!} \left(j^2\right)^{k+i-1} + \frac{1}{2^{k-1}\,k!} \times (-1)^{k+k} \times \binom{2k}{2k} \times \left(k^2\right)^{k+i}$$

$$= \frac{1}{2^{k-1}\,k!} \times \sum_{j=1}^{k} (-1)^{k+j} \binom{2k}{k+j} \left(j^2\right)^{k+i}$$

$$= c_{i+k,k}$$