Expanding the Mandelbrot Set into Higher Dimensions

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Abstract

When in 1980 Benoit Mandelbrot described the $z \rightarrow z^2+c$ formula, many mathematicians and programmers tried to expand the Mandelbrot Set into the third dimension. But all of them where stopped by the non-equivalence in 3D to the 2D complex product $(a+bi) \cdot (c+di)$, something that was well known since times of mathematician W. R. Hamilton. Also, as the 80's computers where not able to produce the calculations needed to represent an image of that kind, all research moved towards other fractal fields. It was in 2007 when the search was recovered by means of a controversial algorithm using algebra based on spherical coordinates triplets $\{\rho, \phi, \theta\}$ (module, longitude and latitude). Although, from a strict mathematical point of view, the process is not correct, the stunning images of the 3D set, especially when raised to higher polynomials $z \rightarrow z^n+c$ soon became an iconic fractal named *Mandelbulb*. The expansion of the Mandelbrot Set in 4D by means of quaternions is also possible. Recent experiments reveal that adequate projecting surfaces provide an infinite group of projections into 3D.

The Origin of the Mandelbrot Set

The origin of the Mandelbrot Set formula may be placed in the early twentieth century when mathematician Pierre Fatou (1878-1929) became interested in the iteration of the equation $c \rightarrow c^2+k$, where c and k are complex numbers with k a constant value. Years after, Gaston Julia (1893-1978) studied how iterating that formula generated a set now known as a Julia Set, whose boundary of infinite length was impossible to draw by hand at that time, even considering it as a finite curve.

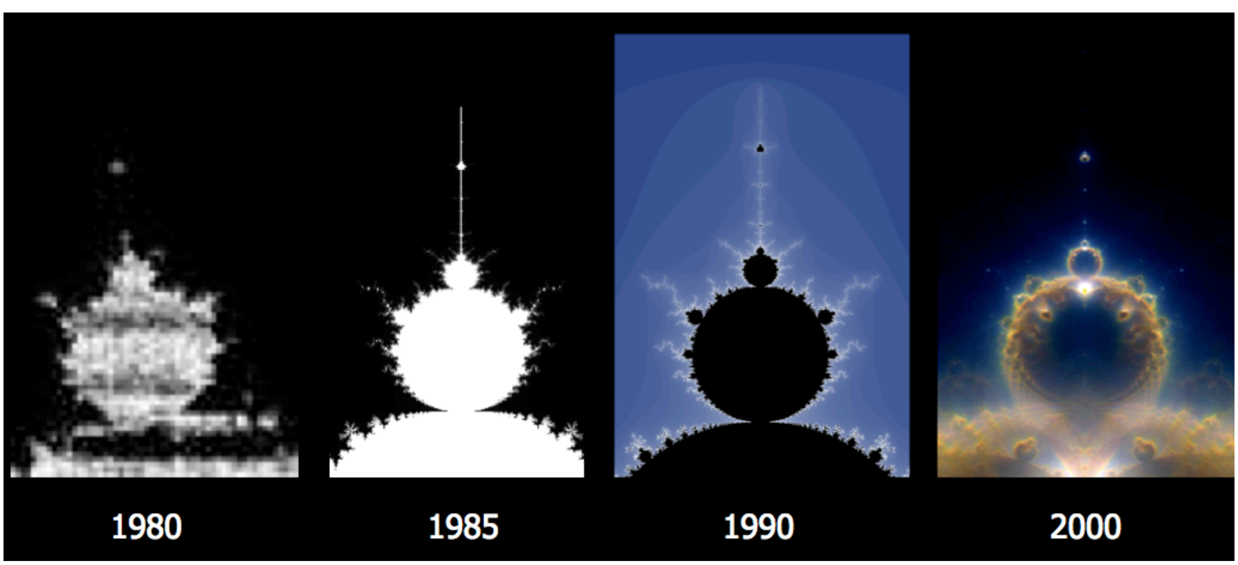


Figure 1: *Graphic Evolution of the Mandelbrot Set between 1980 and 2000.*

The appearance of the Mandelbrot Set on the cover of American Scientist in 1985 coupled with the proliferation of inexpensive personal computers made it enormously popular amongst the BASIC and FORTRAN programmers of that decade.

The emerging interest created under the intricate geometry and enormous complexity of the Mandelbrot Set then turned towards the creation of variants and tweaks of the set [1]. Perhaps the most remarkable was the expansion of the quadratic equation to higher exponents: $z \to z^n + c$.

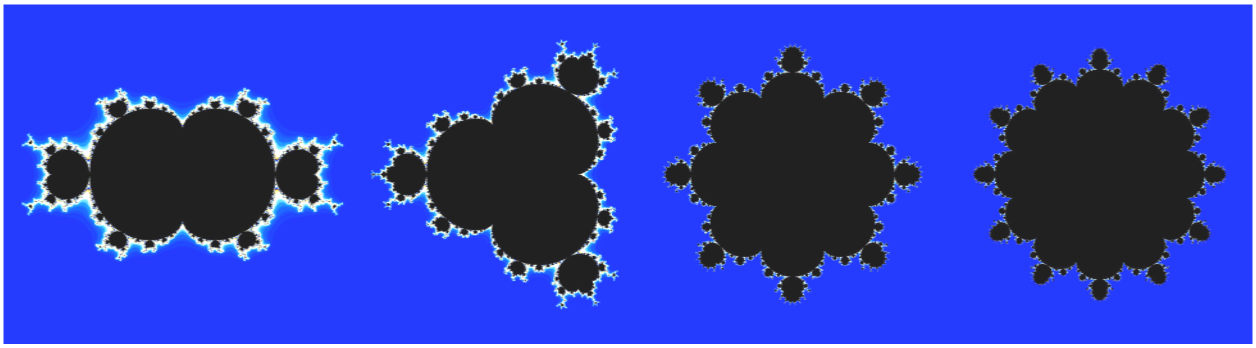


Figure 2: *The Mandelbrot Set variant* $z \to z^n + c$ *for values n=3, 4, 9 and 13.*

Towards a Three-Dimensional Mandelbrot Set

Finding a three-dimensional equivalent of the Mandelbrot Set was undoubtedly one of the obsessions of the mathematicians of that time. But soon that search was cut short by the lack of an Algebra that could operate with a triplet variable $\{a, b, c\}$ in $\mathbf{R}^3$ similarly to the pair of values (a+bi) existing in the realm of complex numbers $\mathbf{C}^2$.

We are able to define the complex variable product (a+bi)·(c+di) :

$$(a, b)\cdot(c, d) = (a\,c - b\,d,\ a\,d + b\,c)$$

But there is no analogous way to multiply two triplets $\{a, b, c\}\cdot\{d, e, f\}$ of real numbers corresponding to two points in 3D space. Nor it is possible to square them, so it is not possible to calculate exactly the equation $z \to z^2 + c$ extended to three dimensions.

In fact, this is nothing new; restrictions in 3D algebras were well known a long time ago. The mathematician Arthur Cayley (1821-1895) demonstrated in the Cayley-Dickson Construction that starting from the body of real numbers one can generate a series of algebras in which each has a dimension double than the previous. That is, there is an algebra for real numbers, with dimension 1, an algebra for complex numbers with dimension 2, an algebra for quaternions, with dimension 4, an algebra for octonions with dimension 8 and so on.

Dimension three is excluded from this sequence and although other algebras exist in three dimensions, they lack the necessary algebraic structure for calculating a Mandelbrot-like Set. In other words, complex numbers inherit the properties of addition and product of real numbers: Commutativity, Associativity, Distributive Law, Neutral Element and Unit Element but there is no analogous algebra for dimension three.

Many mathematicians and programmers gave up this search. Furthermore, personal computers of the era did not have the numerical speed and graphic capabilities to perform the necessary calculations to represent a 3D analogue of the Mandelbrot Set in a reasonable time. But recently the massive thrust of computer graphics has led to some interesting advances in generating and viewing three dimensional fractals or fractal-like structures.

From 2D to 3D: First Approaches

Although the formula $z \rightarrow z^2 + c$ cannot strictly be extended to three dimensions, it is feasible to find other procedures based on the search for alternatives to reproduce it graphically. But here we encounter an important problem; both Fatou and Julia iterated the formula $z \rightarrow z^2 + c$ in order to study its convergence and divergence in the complex field, but never made a description of its geometric characteristics. Mandelbrot publicized the set that now bears his name and sensed its intricate geometry, but he did not specify its construction by means of a purely geometrical process [5].

Later, mathematicians Adrien Douady, John Hubbard, Michal Misiurewicz and John Milnor set some properties permitting the calculations of elements like the main cardioid, disks and bulbs, the basins of attraction, critical points, antennas and small copies of the Mandelbrot Set [3]. These data led us to have an idyllic vision of a 3D version of the Mandelbrot Set, but did not provide enough information to materialize it.

That is why for almost 30 years scientists and fractal enthusiasts hardly experimented with the analogs of the Mandelbrot Set in 3D. However, this did not prevent the more restless programmers from experimenting with certain tweaks in three-dimensional space. Two examples are the height field maps based on the measurement of the potential of the Mandelbrot Set or the projection into the Riemman Sphere. Obviously, despite the spatial appearance of these figures, their calculation was made starting from iteration in the complex plane, later mapped into the 3D space using well-known mapping functions.

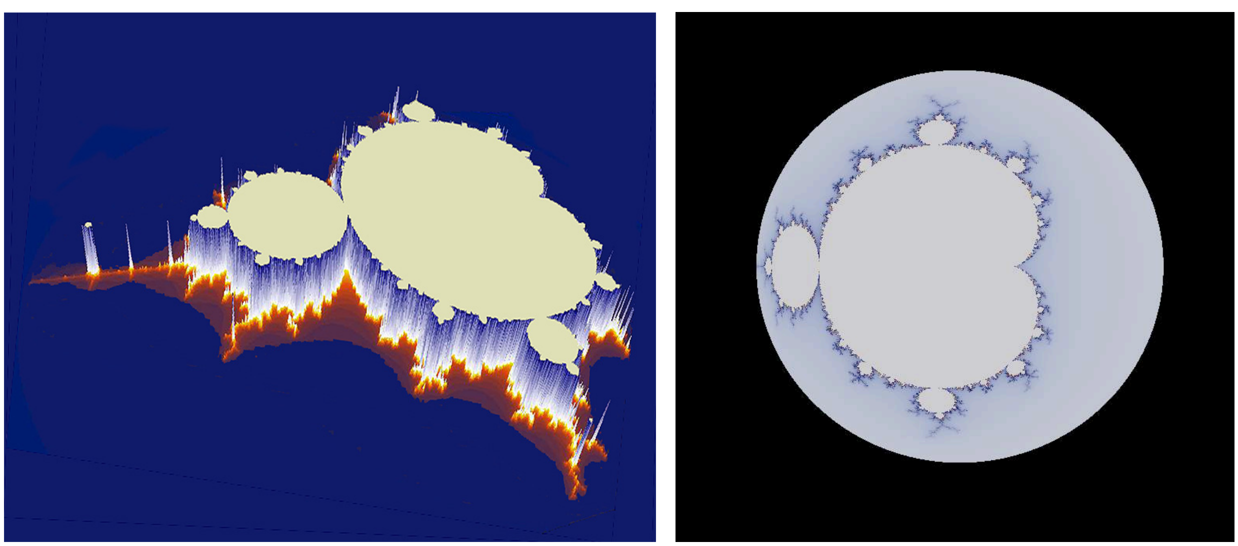


Figure 3: *Two primitive three-dimensional approaches for the Mandelbrot Set. a) Height field based on the potential of the Mandelbrot Set. b) Projection of the complex map into the Riemman Sphere.*

The White/Nylander Formula

In 2007, Daniel White proposed a method to implement the equation $z \to z^2 + c$ into three dimensions. It was based on observing the geometrical behavior of the product in one and two dimensions so as to extend it successfully into three dimensions. More specifically he was interested in squaring the variable z^2, which was proved to be impossible by means of analytic ways [6].

Let us imagine the one-dimensional squaring operation as a mapping that *stretches* a distance along a line, when values are greater than 1 and *shrinks* it, when values are less than 1. The case of the product in the two-dimensional complex field is more sophisticated, as it is based on rotation. Let us suppose we square any complex number, for example 0.2+0.3i :

$$(0.2 + 0.3\,\mathrm{i})^2 = -0.05 + 0.12\,\mathrm{i}$$

If we find the angle with *X* coordinate axis and the modulus of number (0.2+0.3i) :

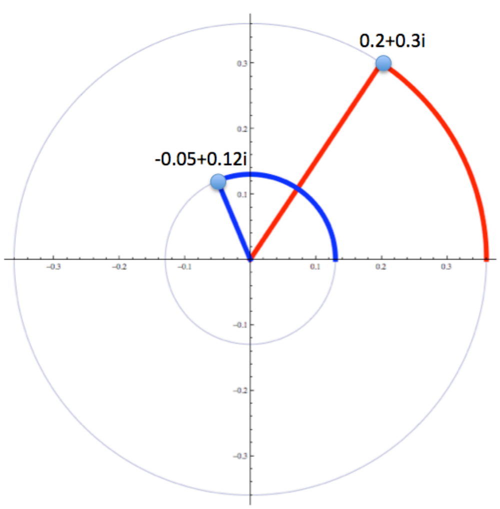
$$\mathit{angle} = \mathrm{ArcTan}[\,0.3 / 0.2\,] = 56.31^{\circ}$$
$$\mathit{modulus} = \mathrm{Sqrt}[\,0.2^2 + 0.3^2\,] = 0.36$$

we verify that the solution (−0.05+0.12i) is a complex number composed with a rotation of 112.62º, twice the angle of 0.2+0.3i (2 × 56.31º) and modulus 0.13, square of the modulus of 0.2+0.3i (0.36^2).

From this observation, White suggested that the rotation that appears when squaring a complex number could be extended to three dimensions using the spherical coordinates as an equivalent expansion.

The spherical coordinates use a modulus ρ and two angles: longitude or azimuth represented by ϕ and latitude or elevation angle, represented by θ (note that in some literature symbols are swapped). Thus, the triplet value for the three-coordinate system is represented as $\{\rho, \phi, \theta\}$ and squaring is defined as

$$\{\rho, \phi, \theta\}^2 = \{\rho^2, 2\phi, 2\theta\}$$



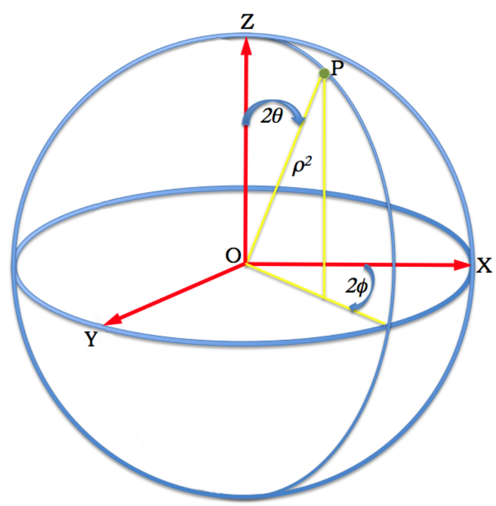


Figure 4: *a) The squaring of the complex number 02+0.3i is expressed geometrically as a rotation doubling the angle and squaring the modulus. The result is -0.05+0.12i. b) White proposes that the equivalent of this process in 3D is the composition of two rotations expressed in spherical coordinates. As it happens in 2D, the rotation angle (ϕ and θ) is doubled and the modulus is squared.*

Since there is a well-known relationship between Cartesian and Spherical coordinate systems, we may define the squaring of a Cartesian triplet by calculating its modulus, ρ, and angles, ϕ and θ.

$$\rho = \mathrm{sqrt}(\ x^2+y^2+z^2\ ),\quad \theta = \arctan(\ y/x\ ),\quad \phi = \arcsin(\ z/\rho\ )$$

$$\{\ x, y, z\ \}^2 = \rho^2\ \{\ \cos(2\theta)\cos(2\phi), \sin(2\theta)\cos(2\phi)\ , -\sin(2\phi)\ \}$$

Triplet addition is no problem since it follows the Cartesian system:

$$\{x, y, z\} + \{c_1, c_2, c_3\} = \{x+c_1, y+c_2, z+c_3\}$$

Then we can easily generate a formula to reproduce in 3D space the Mandelbrot Set $z \rightarrow z^2+c$. The final equations have been polynomialized to improve the performance of computer programs:

$$\mathrm{newx} = c_1 + x^2 - y^2 - z^2$$

$$\mathrm{newy} = c_2 + 4xyz\ /\ \mathrm{sqrt}(\ y^2 + z^2\ )$$

$$\mathrm{newz} = c_3 + 2x\ (\ -y^2 + z^2\ )\ /\ \mathrm{sqrt}(\ y^2 + z^2\ )$$

Fortunately, the Euclidean distance behaves in a very similar way under 2D and 3D, so the divergence condition to break the iteration $|z|>2$ may be easily referred as:

$$\mathrm{sqrt}(\ x^2 + y^2 + z^2\ ) > 2$$

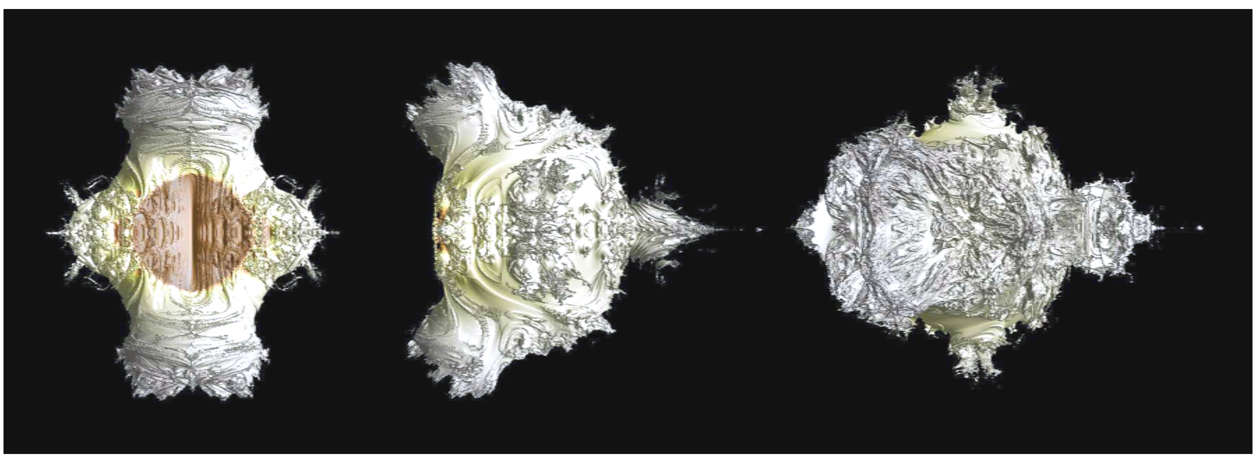


Figure 5: *Three views of the Mandelbrot Set in 3D seen from (1,0,0), (0,1,0), (0,0,1).*

So, Is It Really the 3D Mandelbrot Set?

Not really. The plots do not produce the kind of results fractal experts expected to find. There are things that seem to work well, as the plane $z = 0$ containing the 2D Mandelbrot set, But in some areas of the set there's a huge lack of detail while other areas are too messy. Also the location of some bulbs looks unnatural.

However, despite its limitations, the 3D Mandelbrot Set has received an overwhelming response from the fractal community, especially from [6] under the moderation of David Makin. Much of the success of this new 3D Mandelbrot is due to the work of Paul Nylander, who improved Daniel White's formula and proceeded to define with rigor the algebraic properties necessary to confirm his formula as the top candidate to be acknowledged as 3D Mandelbrot Set.

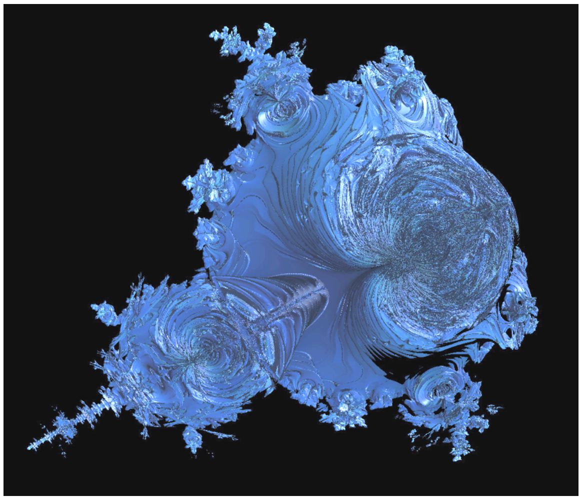
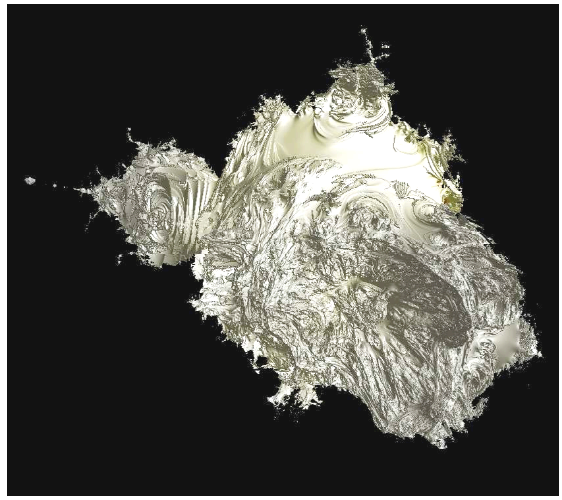



Figure 6: *Two different versions of the White / Nylander formula for the Mandelbrot Set in 3D.*

But from a rigorous point of view there are several reasons to disqualify the candidacy of this formula to be the 3D expansion of the Mandelbrot Set. First, the original Mandelbrot Set was defined in 2D, so it is only a flat figure. Its author, Mandelbrot has never published any detailed reference to its 3D extension and unfortunately there is not enough geometrical information in the 2D formula to generate a 3D analog.

Second, the 3D expansion of the product is not sufficiently justified. If we understand the *1D* product as a *stretching* and the *2D* product as a *rotation*, it appears that *3D* product would be something like a *super-rotation*. But the non-commutative composition of two rotations doubling its angles and squaring the modulus seems to be a forced solution for an unanswered problem.

Finally, there are many steps in which subjective decisions are taken. For example, by applying rotations in different order, from different axes and different angles, different results are given. Although all rotation formulae are mathematically correct, they do not strictly represent the 3D Mandelbrot Set, just a similar but different thing. In many cases, processes like the distance estimation method or the coloring techniques were taken on purely aesthetic criteria, helping the survival of the more beautiful set, but not necessarily the correct one.

The Mandelbulb

Another of the successes of Paul Nylander was his interest on the 3D expansion of equation $z \rightarrow z^n + c$, as it was made 20 years ago in 2D complex formula. Thus the triplet $\{\rho, \phi, \theta\}$ is raised to n^{th} power:

$$\{\rho, \phi, \theta\}^n = \{\rho^n, n\phi, n\theta\}$$

$$\{x, y, z\}^n = \rho^n \{\cos(n\theta)\cos(n\phi), \sin(n\theta)\cos(n\phi), -\sin(n\phi)\}$$

Given the limited appeal of the Mandelbrot Set quadratic formula in 3D there were no high hopes placed on plots with other exponents but, surprisingly, there appeared a new iconic fractal type called *Mandelbulb*, looking similar to an asteroid of rich structure and great detail.

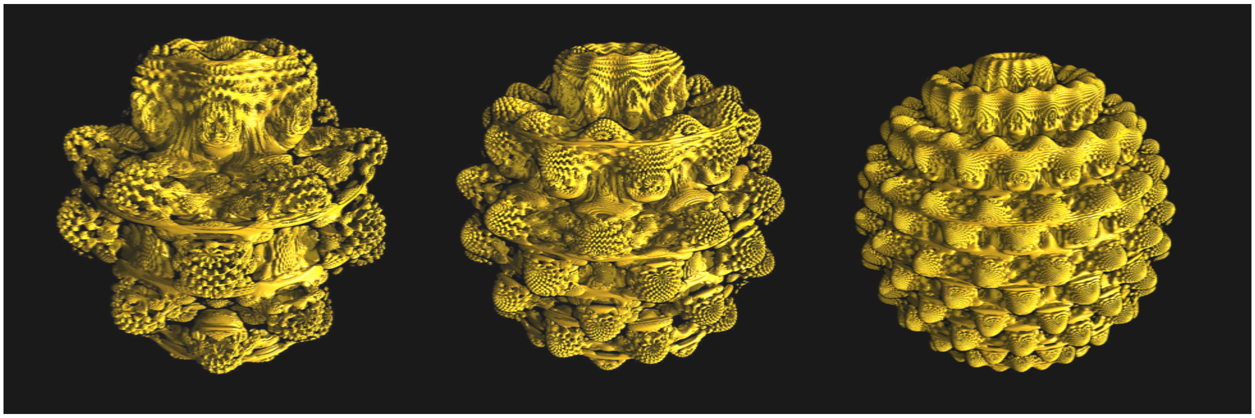


Figure 7: *Mandelbulbs with formulae* $z \rightarrow z^8 + c$, $z \rightarrow z^{12} + c$ *and* $z \rightarrow z^{20} + c$.

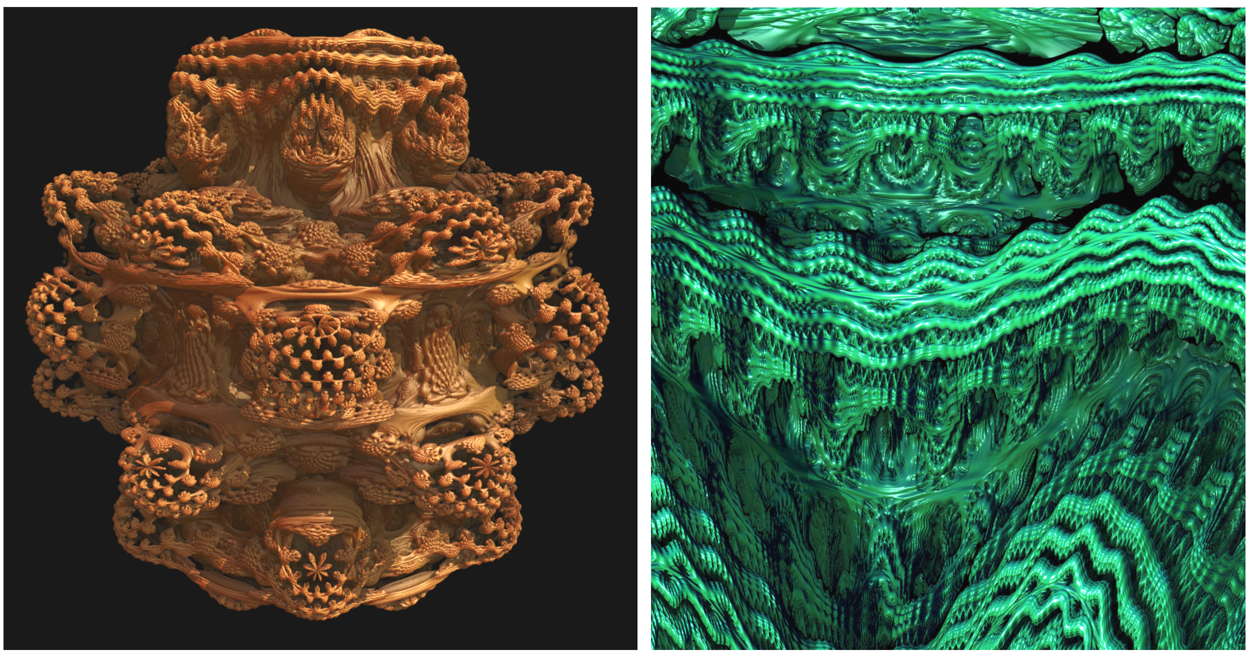


Figure 8: *a) Mandelbulb* $z \rightarrow z^8 + c$ *b) Surface detail of* $z \rightarrow z^{12} + c$.

The Mandelbrot Set in 4D

William Rowan Hamilton (1805-1865) spent part of his life trying to find a three-dimensional analog for complex numbers [2]. After many disappointing attempts he conceived the idea of quaternions, a natural extension of complex numbers in 4D. A quaternion is expressed as $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ satisfying the equation $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{i\,j\,k} = -1$. Quaternions inherit from complex numbers all its properties, except that quaternions product is non-commutative, for example $q_1 \cdot q_2 = m$, while $q_2 \cdot q_1 = -m$.

The general formula for Julia fractals works perfectly in 4D by means of quaternions producing a wide range of fractal shapes (Figure 9a). Orthogonal projection on 3D space is made easily by simply ignoring one dimension [4]. Surprisingly, when this technique is applied to the 4D Mandelbrot Set, the result is a disappointing figure based on the revolution along *X* axis of the 2D Mandelbrot Set (Figure 9b).

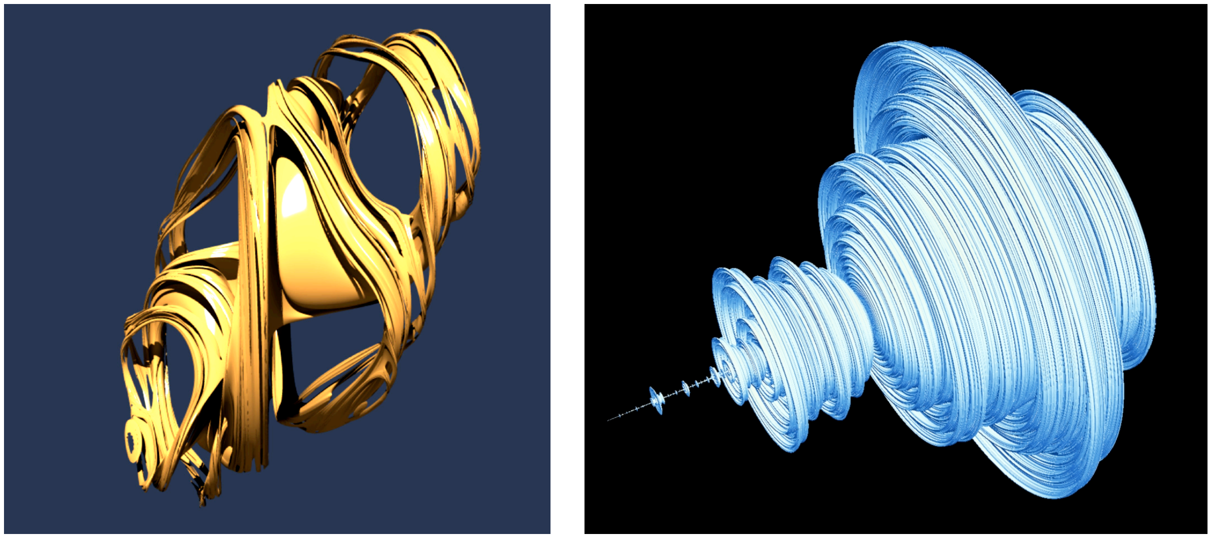


Figure 9: *a) Quaternion Julia Set b) Quaternion Mandelbrot Set.*

Ongoing research in the University of the Basque Country confirms that the apparent lack of interest shown in Figure 9b is due to the wrong choice of projection surfaces to create visual interesting 3D *slices* of the 4D Mandelbrot Set. Making one dimension equal to zero produces a very simple projection of the 4D Set into 3D (Figure 9b). As an example, imagine projecting a 3D cube into 2D by looking it from a viewpoint perpendicular to one of its faces; the result will be a square. So, other projecting planes are required instead of just making null one dimension.

Making an exhaustive computation of the four variables $\{i, j, k, w\}$ of the quaternion space and projecting the 4D Mandelbrot Set over a basic 4D plane of equation $ai+bj+ck+dw=0$, it has been proved that there is more activity than reported in the fractal forums, as shown in the projections of Figure 10. The main problem by introducing the fourth dimension in the calculation is the enormous increase of processing needs and the difficulties of selecting an adequate projecting surface.

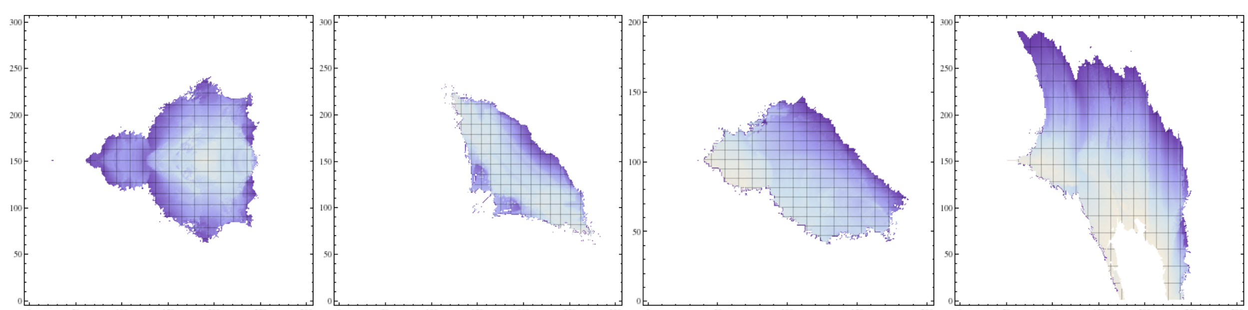


Figure 10: *2D projections of the 4D Mandelbrot Set by performing a four variable computing.*

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