Oddity of layered spirals

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Attachment to the paper 'Fractional Beauty', included in the Proceedings of the 2014 Bridges Conference.

Subject of my conference paper is how to create beauty by means of continued fraction, shortly discussing two approaches: one resulting in stripe patterns, the other one resulting in crystal like patterns. In this attachment more light is shed on the first approach. It describes the beauty and elegance, also with regard to the underlying arithmetic/geometric concept, of layered angularly winding spirals that can be generated by the introduction of lines of deflection within plane symmetries.

Plane symmetries. There are 17 different diagrams for symmetrical repetition of a motif in the plane (Figure 1). The formal code table is presented in Table I.

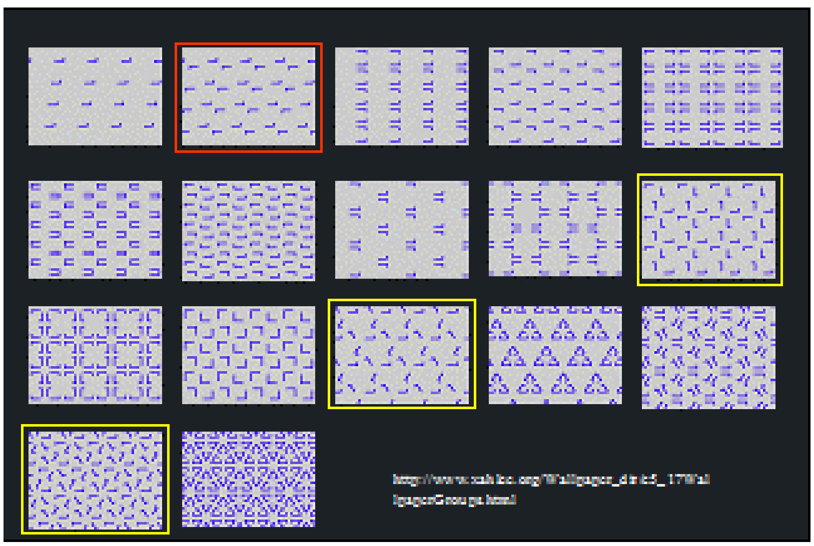

Figure 1: *17 different plane symmetries*

p1	p2	pm	pg	pmm
pmg	pgg	cm	cmm	p4
p4m	p4g	p3	p3ml	p3lm
p6	p6m			

Table I: *Formal code table of the seventeen plane symmetries.*

Each of these seventeen is composed out of one or more of four 'spatial transformation': translation, rotation, reflection and glide-reflection (Figure 2). These transformations can be conceived as 'symmetrical operators' which are 'active' in de plane.

Figure 2: *For spatial transformations: translation, rotation, reflection and glide-reflection.*

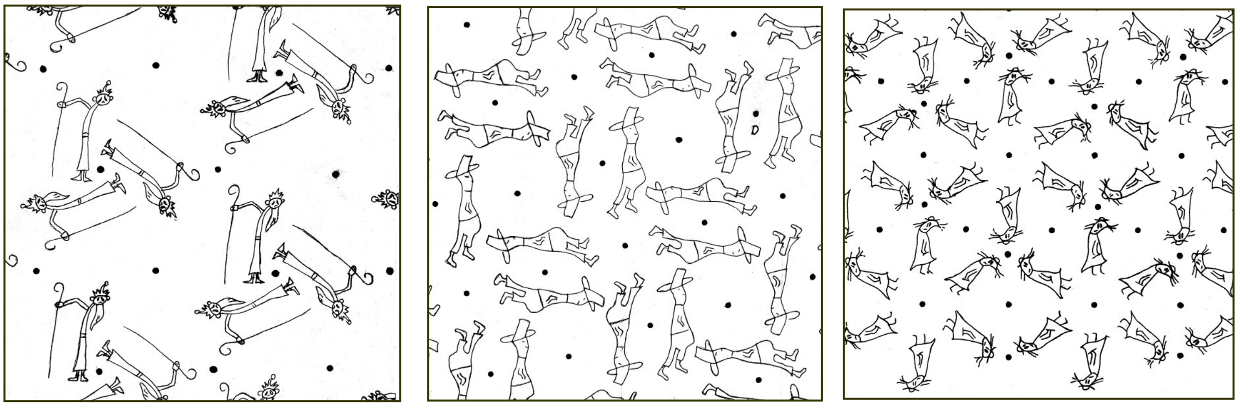

Figure 3: *Schedule of spreading of axes of rotation through the plane in p3, p4 and p6.*

Especially suitable for the design of spirals are plane symmetries in which only rotations occur beside shifts. There are four of them: p2, p3, p4 and p6 (marked through a yellow frame in Figure 1). We focus on the last three, which are characterized by a wonderful schedule of dispersion of the axes of rotation through the plane (Figure 3). Three different types of axes occur in each. In p6 the difference between these three types is directly related to the 'foldness': there are 2-fold, 3-fold and 6-fold axes. In p4 this partly applies. There are 2-fold and 4-fold axes. Within the fourfold axes two different types occur, one type lying between the feet of the males and the other one at breast height. In p3 all axes are threefold: one type lying between the crowns, one between the feet and one between the staffs of the kings. In p6 and p4 the axes with lower 'foldness' occur in different spatial orientations[1].

Stripe paths. We can introduce lines of deflection of different lengths in each of the three plane symmetries, in such way that each of the three types of rotation axes is linked to its 'own' pair of deflection lines as to their length value. This results in a division of the plane in repeat units, each of which can be divided in stripes (Figure 4). Stripes in different repeat units unite which results in 'stripe paths' with the nature of closed circuits. The part of a deflection line which is traversed by the path is the unit of length. The real length of this unit increases as the angle of bending

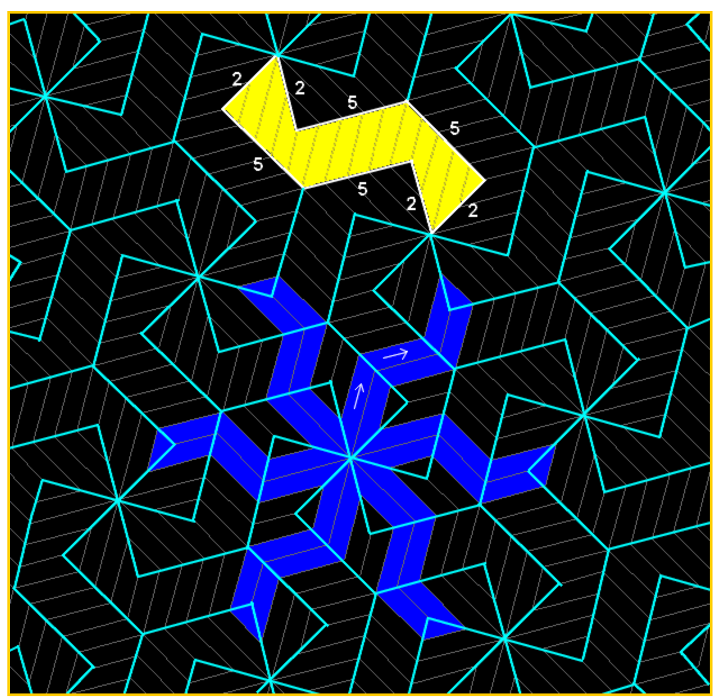

Figure 4: *The length ratio of the lines of deflection determine the course of stripe circuit.*

[1] In p6 the 3-fold axes occur in two different spatial orientations and the 2-fold axes in three. In p4 the 2-fold axes occur in 2 different spatial orientations.

becomes sharper[2]. The introduced length ratio determines the course of the stripe path. In principle stripe paths are layered spirals. That may not always be obvious when the values in the length ratio are small, as for example in Figure 4, but when we enlarge them, wonderful spiral winding may emerge (Figure 5).

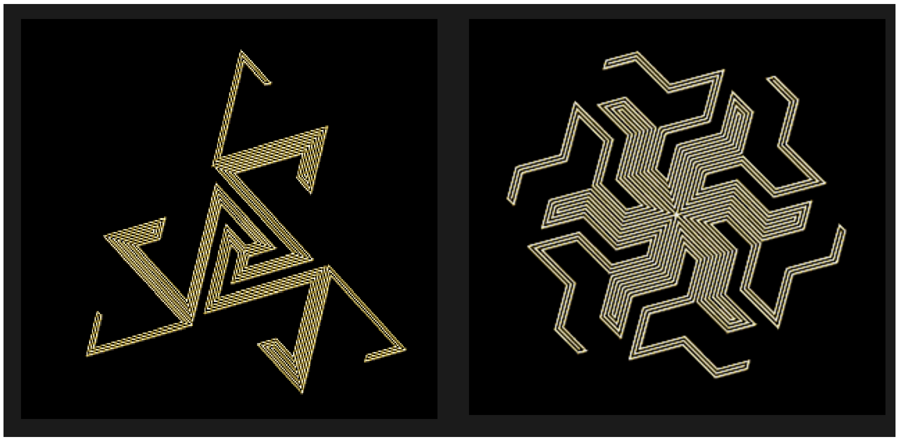

Figure 5: *When values in the ratio become larger, wonderful spiral structures may arise.*

Figure 4 shows a simplified version of the system of deflection lines in p6[3]. The complete system of deflection lines for each of the three plane symmetries is shown in Figure 6. It has the nature of a repeat unit. The three units are broadly similar, with exception of the angles at which deflection lines meet each other. In the perimeter of the units there are three types of deflection lines which are pairwise connected to the three types of rotation axes. Each pair has its own length value. These length values, which we call S(mall), M(edium) and L(arge), are mutually variably in a restricted way. The interdependency is specified by the equation M = L-S.

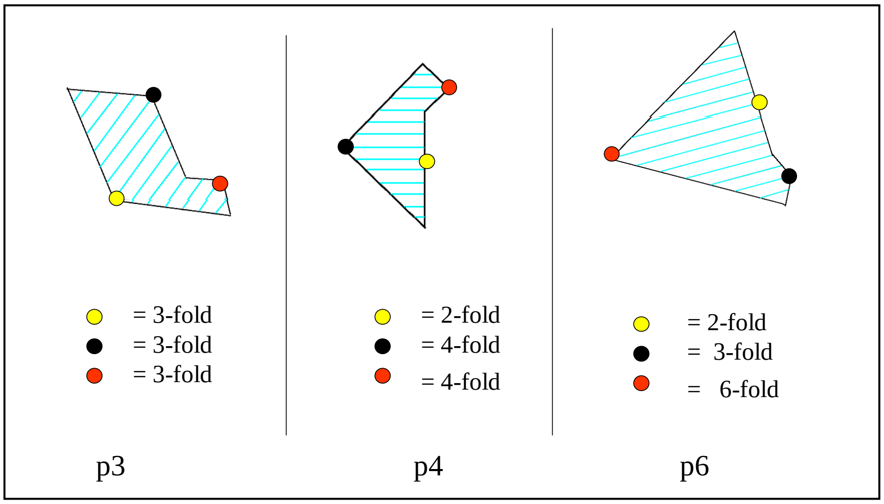

Figure 6: *System of deflection lines for each plane symmetry.*

[2] In this example, the unit is lager by bending at angles of 60° than by bending at angles of 120°.

[3] The pair of deflection lines which connects 2-fold axes is not shown in this picture. See Figure 11 for a full view of the system of deflection lines in p6 , including the lines which connect 2-fold axes.

As to the linking of the different lines of deflection (with length values S, M and L respectively) to the different axes of rotation, the number of options differs per plane symmetry. In p3 there 's but one option; in p4 there are two and in p6 there are 4 options (Table II).

p3

3-fold	S
3-fold	M
3-fold	L

p4

2-fold	S	M
4-fold	M	S
4-fold	L	L

p6

2-fold	S	S	M	M
3-fold	M	L	S	L
6-fold	L	M	L	S

Table II: *Possible linking of S , M and L to the different axes of rotation*.

Every stripe path can be represented by a 'composite' ratio that we denote as S/M/L and contains the ratio's S/L, S/M and M/L.

Reversion of acute and obtuse. The system of deflection lines in p6 is remarkable. When we mutually exchange the length values of the deflection lines which are connected with the 3-fold and 6-fold axes of rotation, a reversal occurs in the ratio between acute and obtuse angles in the perimeter of the stripe path. So every stripe path has its reverse as to the ratio between acute and obtuse[4] (Figure 7).

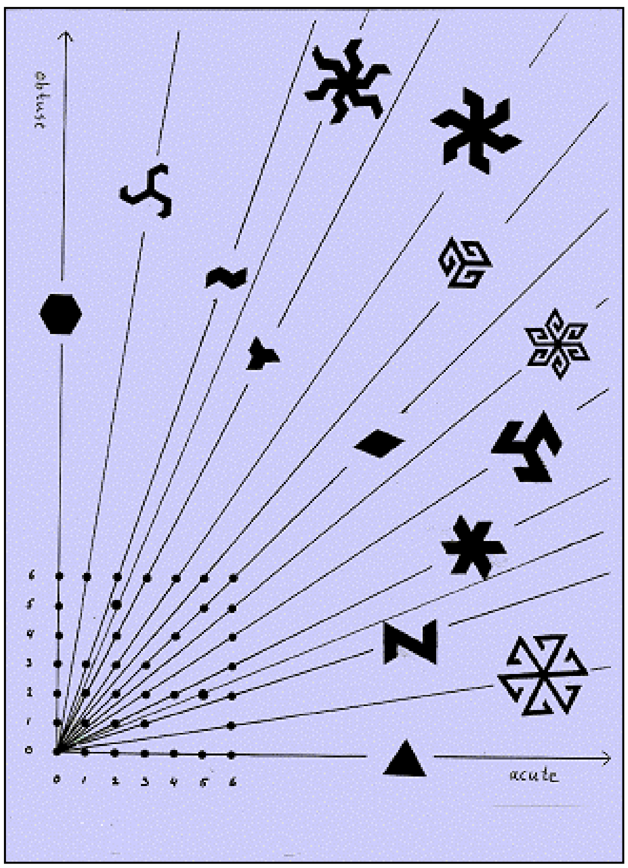

Figure 7 : *In p6, every stripe path has its reverse as to the ratio between acute and obtuse angles.*

4 Moreover, this implies also changes in the type of rotation axis which lies in the center of the stripe path: 6-fold axes change in 3-fold and vice versa. However, 2-fold axes in the center don't change in type with a reversion of the acute/obtuse ratio.

Perimeter stripe path build up of spirals. A stripe path always has a rotation symmetry, the 'foldness' of which is defined by the axis of rotation which lies in its center. Axes of the other two types lie on the perimeter of the stripe path. How many axes of each lie there, depends on the 'fold-ness' of the rotation symmetry of the path. When it has for example a 4-fold symmetry, from each of the other two types there lie four on the perimeter, dividing it in eight uniform parts (Figure 8). Each part has the character of an angularly winding spiral. So the perimeter can be conceived as build up out of spirals that are coming from each other by alternately rotating them around the one and then around the other type of rotation axis

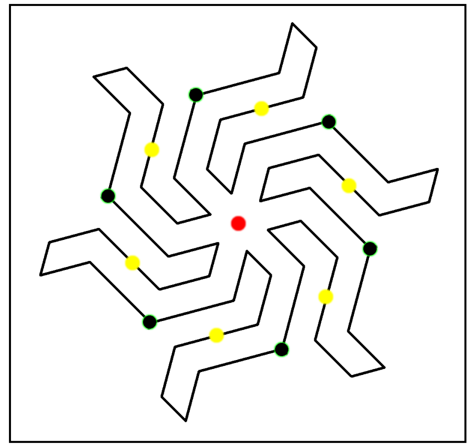

Figure 8 : *Each of the for parts in the perimeter represents the course of the spiral.*

Basic stripe paths. For a good understanding of the ideas that are unfolded in the following, we need to reflect in some detail about the most basic stripe paths, for which the composite ratio is 0/1/1 or 1/1/0 or 1/0/1. These ratios expresses that in the system of deflection lines one of the three length values has reduced to zero. That means that the respective axis or rotation no longer has its 'own' pair of deflection lines, but lies on the intersection point of the pairs of deflection lines of the other two axes of rotation. In Figure 8 we showed that the perimeter of a stripe path can be conceived as resulting from a series of rotations of a spiral around alternately the one and then of the other type of rotation axes. This also applies to the basic stripe path, be it in this case the spiral is in a sort of prenatal stage, having the profile of a straight line (Figure 9).

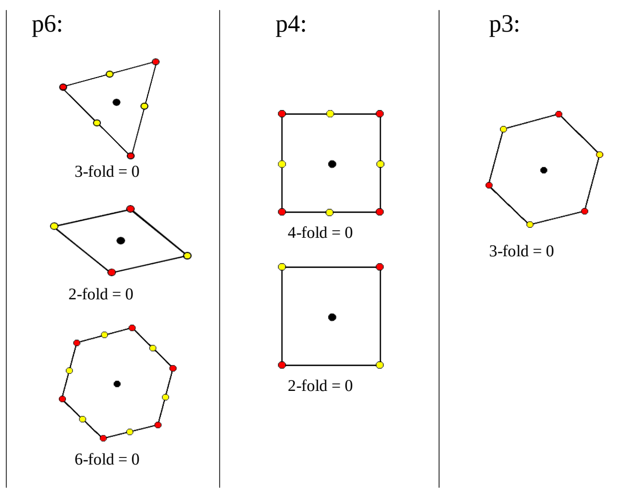

Figure 9 : *In the basic stripe paths, the spiral has the profile of a straight line.*

Series of S/M/L-events. A stripe path in its course generates S- , M- and L-events, when crossing the respective deflection lines. So each stripe path can be conceived as a series of S/M/L-events. The crossing of a deflection line with length value S is always directly followed by the crossing of a deflection line with length value L. So we can conceive the combination of these two crossings as a basic element (SL) in the S/L/M-series. Also the crossing of a deflection line with value M is always directly followed by the crossing of a deflection line with value L. So the combination of these two crossings also can be conceived as a fixed element (ML) in the S/L/M-series.

Figure 10 shows the S/M/L-series for the composite ratio 5/12/17. Such a series always has a layered character which can be represented in a structure diagram (Figure 11). The diagram focuses on the S- and L-events in the series[5]. At each layer there is an alternation of *n* and *1*. In the first layer, these elements indicate the number of L- events between two S-events, being always *n* or *n* + *1*, whereby *1* is

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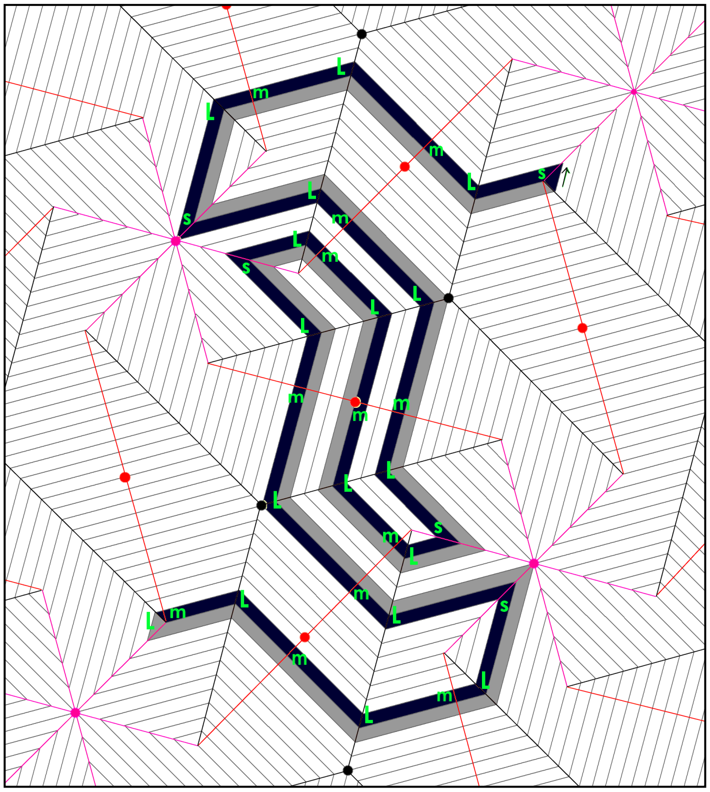


Figure 10: *Series of SML-events.*

recorded as a separete element besides *n*. After the first layer, the alternation of *n* and *1* continues, but now these elements indicate the number of *n*- elements between two *1*- elements at the preceding level. Over the different layers, these patterns of alternation of *n* and *1* realize the best solution for an even spreading of S- events on L-events. The value *n* for the different layers can be derived from the ratio S/L[6] by applying a proces of repeated division to it. In the first division you divide L by S. In the following divisions you divide the divisor of the preceding division (becoming the dividend now) by its the remainder (becoming the divisor now), until the remainder is 0. The result is a series of partial quotients which are indicated as a_1, a_2 ,a_3, .. In the case of the composite ratio 5/12/17 the series of *a*-values of S/L is: a_1= 3 , a_2 =2, a_3 =2. Figure 11 shows that the structure diagram perfectly matches with this series of *a*-values: the number of layers in the structure diagram corresponds with the number of *a*-values in the

[5] Because SL and ML are fixed elements in the series, we don't neglect aspects of complexity bij focussing on S- and L-events in the structure diagram.

[6] The continued fraction can also be applied to the ratios M/L and S/M . However, the continued fraction of the ratio S/L is most directly related to the layered structure of the spirals .

continued fraction; the successive values for *n* in the structure diagram corresponds with the successive values for *a* in the continued fraction.

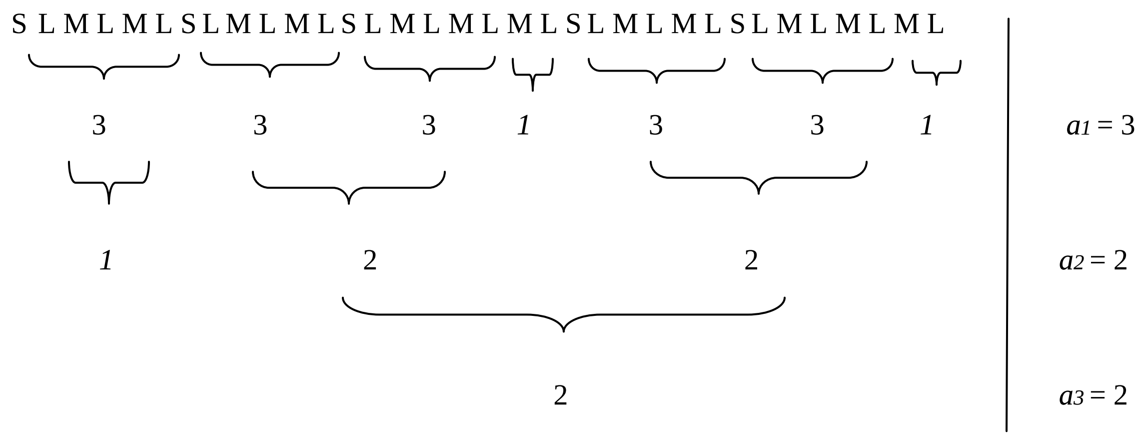


Figure 11: *The relation between the layers in the SML-series and the a-values in the continued fraction .*

Majors and minors. The layered structure of the S/M/L- series can be transformed in a fragmentation scheme in which simple composite ratios are aggregated to more complex (Figure 12). The simplest are

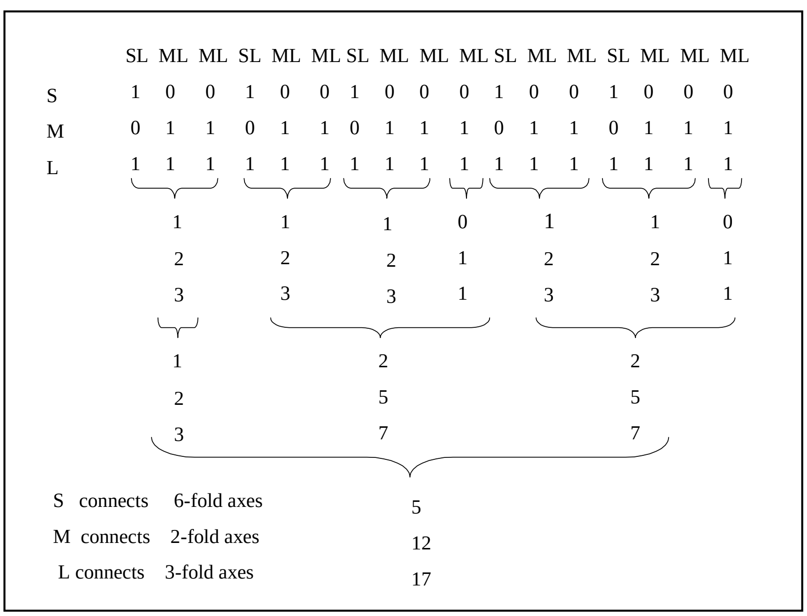


Figure 12: *Layered aggregation of simple composite ratios, resulting in 5/12/17.*

1/0/1 and 0/1/1 or 1/0/1 . They are the 'building blocks' in the building up of the finale composite ratio. The difference between M and L has disappeared at this level. At the following levels, aggregation of the simple composite ratios takes place according to the layered pattern of alternation in terms of *n* and *1* as discussed above.

Minors and majors. In every layer in the diagram in Figure 12 there are composite ratios which are in the majority and composite ratios which are in the minority. We call them 'majors' and 'minors' respectively. The majors at a certain level are aggregates of one minor and a number of majors at the preceding level. That number equals *n* at the present level. And the minors at a certain level are majors at the preceding level. In particular those majors which are recoded as *1* at the present level (Figure 13).

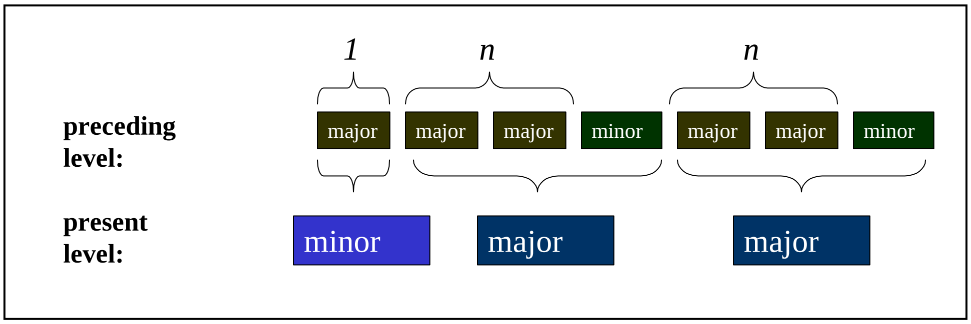


Figure 13: *Majors and minors at a certain level are derived from the preceding level.*

At each level the major and minor both represents a stripe path. These stripe paths we also call 'majors' and 'minors' respectively.

Layers in spiral formation. To understand how the successive layers in spiral winding are structured, the concept of 'majors' plays a central role. Let us take as example the stripe path of which the composite ratio is 13/56/69. The pattern as a whole, and one of the stripe paths from which it is built up, are shown in Figure 14.

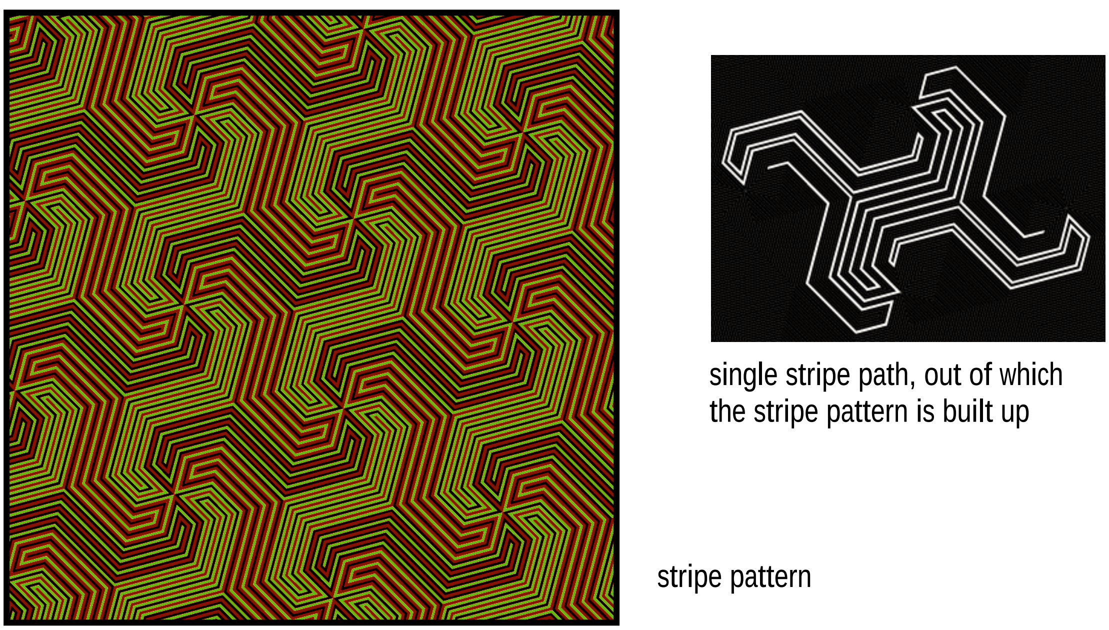


Figure 14: S*tripe pattern and single stripe path corresponding to 13/56/69.*

The *a*-values in the continued fraction of 13/69 are: $a_1 = 5$, $a_2 = 3$ and $a_3 = 4$. The majors of the three layers in spiral winding are shown in Figure 15. They are derived from the series of *a*-values by successively dropping the last value in that series (Table III). As to the major of the first layer, there are no preceding *a*-values in the series. Here the basic stripe path (see Figure 9), representing the pre-spiral in stage, delivers the perimeter for spiral winding. In this case that is the hexagon. As Figure 15 shows, in each layer in the building up of the final stripe path, spiral winding takes place around the perimeter of its major.

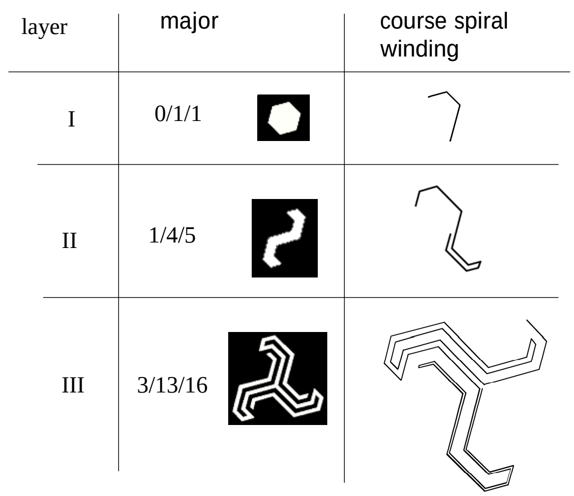


Figure 15: *Building up of layered spiral winding within the perimeter of stripe path 13/56/69.*

series of *a*-values	composite ratio majors	layer
5 3 4		
5 3	3/13/16	III
5	1/4/5	II
-	0/1/1	I

Table III: The *composite ratios of majors can be derived from the series of a-values.*

Number of spiral windings per layer. The value *a* of each layer indicates how far spiral winding goes through in that layer. In Figure 16 this is shown for the first two layers in spiral winding in P3. The hexagon is the major in the first layer and spiral winding takes place around the perimeter of this polygon. If we gradually enlarge the value of a_1 (most left column), there arises more and more spiral winding 'around' the perimeter of it. Each of the stages in spiral winding in this first layer delivers a major for spiral winding in the next layer. If we enlarge value a_2 for each of these stages, more and more spiral winding takes place around the perimeter of the respective major.

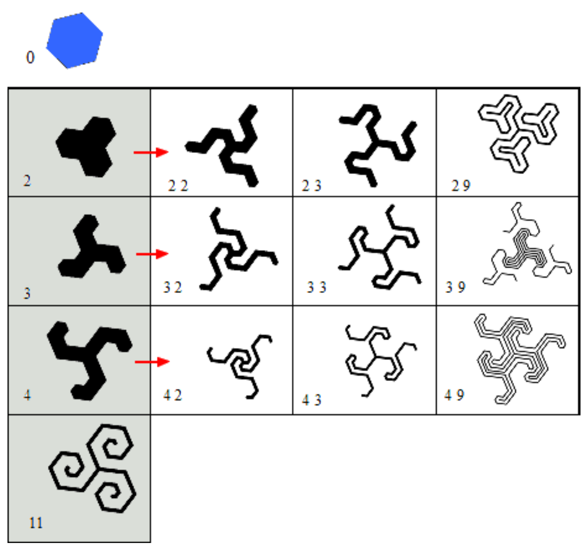
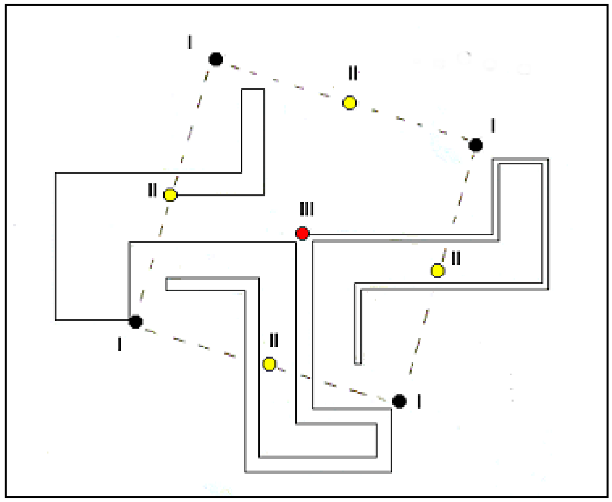


Figure 16: *The value a of a layer indicates the degree to which spiral winding carries on in that layer.*

Centers of spiral winding. The center of spiral winding at a certain level is the rotational axis which lies in the center of its major. When there are several layers in a stripe path, at each layer one of the three types of axes of rotation is the center of spiral winding. In the example beneath (Figure 17) this is illustrated for the composite ratio 16/37/53 in P4, which has 3 layers. Each type of rotation axes can be center of spiral winding in more than one layer.



$a_1 = 3$
$a_2 = 3$
$a_3 = 5$

Figure 17: *Different types of rotation axes as center of spiral winding in three layers.*

When one value *a* = 1. In the preceding explanation we have talked about series of *a*-values in which all the *a*-values are greater than 1. When at any moment in the series of *a*-values[7] a value *a* is 1, this causes that in the preceding layer *n + 1* is in the majority instead of *n*. In such cases, the value *n* in the next layer indicates the number of *(n+1)* elements between two *n* elements in the preceding. Below (Figure 18) this is illustrated for the composite ratio 3/11/14. The series of *a*-values in the continued fraction of 3/14 is: $a_1 = 4$; $a_2 = 1$ $a_3 = 2$. The value a_2 does not represent an own layer.

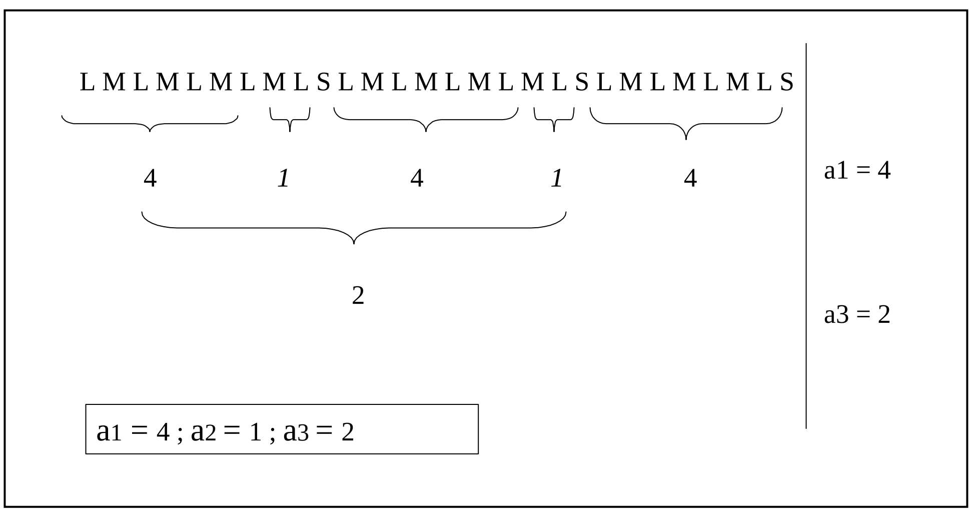


Figure 18: *When a=1 at the preceding layer a+1 is in majority.*

When more values a in succession are 1. When more *a*-values in succession are 1, the first of these, as just explained , causes that *n + 1* is in the majority instead of *n*, in de preceding layer. It doesn't represent an 'own' layer. The second behaves again 'normally', indicating that the number of *n's* between

[7] The first *a*-value is never 1 because of the interdependancy equation M = L-S

1 in the respective layer is 1. The third again behaves deviant, causing majority of $n + 1$ in the preceding layer, the fourth again behaving normally, etcetera. When one or more values *a* are 1 in the series of *a*-values, the derivation of the successive majors needs some adjustment, because in such case we cannot always simply let drop successive *a*-values. That is because the last value in the series of *a*-values always needs to be larger than 1[8]. When the last *a*-value after canceling a previous one becomes 1, it must be added to the, at that time penultimate *a*-value. To elucidate this let us take the composite ratio 21/34/55. The series of *a*-values of the ratio S/L= is 2 1 1 1 1 1 2. When we diminish it conform the rules above, 4 composite ratios can be derived, which represent the major of the successive layers (Tabel IV).

series of a-values	**composite ratio majors**	**layer**
2 1 1 1 1 1 2		
2 1 1 1 2	8/13/21	IV
2 1 2	3/5/8	III
3	1/2/3	II
-	0/1/1	I

Tabel IV: *Deriving the composite ratios of the majors at 4 levels.*

Figure 19 shows the stripe pattern generated by 21/34/55 and the majors at the 4 levels. These majors can easily be recognized in the overall course of the spiral winding.

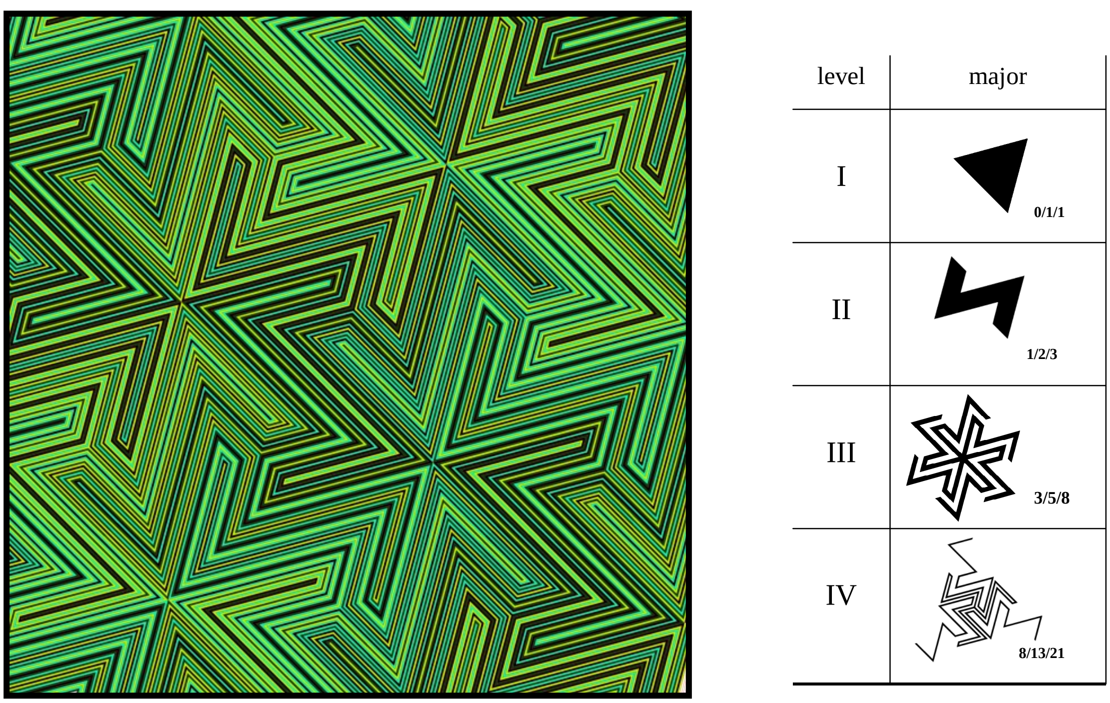


Figure 19: *Stripe pattern generated by 21 /34/55 and the majors at each of the 4 levels in spiral winding.*

[8] Except when the series is diminished to the first a-value which is always 1.This value 1 represent ast stated before the pre-spiral level and represent the basic stripe path, which is a square or a triangle or a hexagon or a rhombus or a trapezium, etc..

Color mixing. The potentials for color mixing makes the design of angularly twisting spirals challenging. The most simple way to get color mixture is: introduce *x* different colors in the pattern and create spiral formation around a spiral center which realizes a color mixing scheme of *y* out of

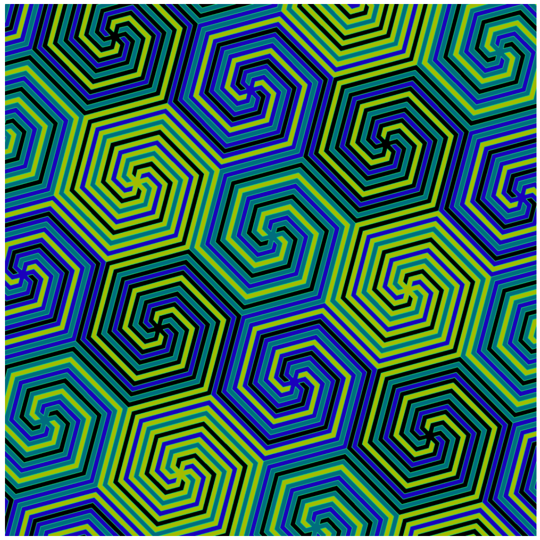


Figure 20: *Three out of four colors mix around each center of spiral twisting*.

those *x* colors, whereby *y* is smaller than *x*. Such as, for example, in the pattern in Figure 20. Around each center of spiral winding 3 out of the 4 colors are involved in the winding .

Layered color mixing. But more intriguing are the stratified color mixtures. Let us for example look at the pattern in Figure 21a.The composite ratio here is 12/29/41 The major at each level is shown in Figure 21b. Color mixture takes place at level I and III. At these levels the major has the same axis of rotation in

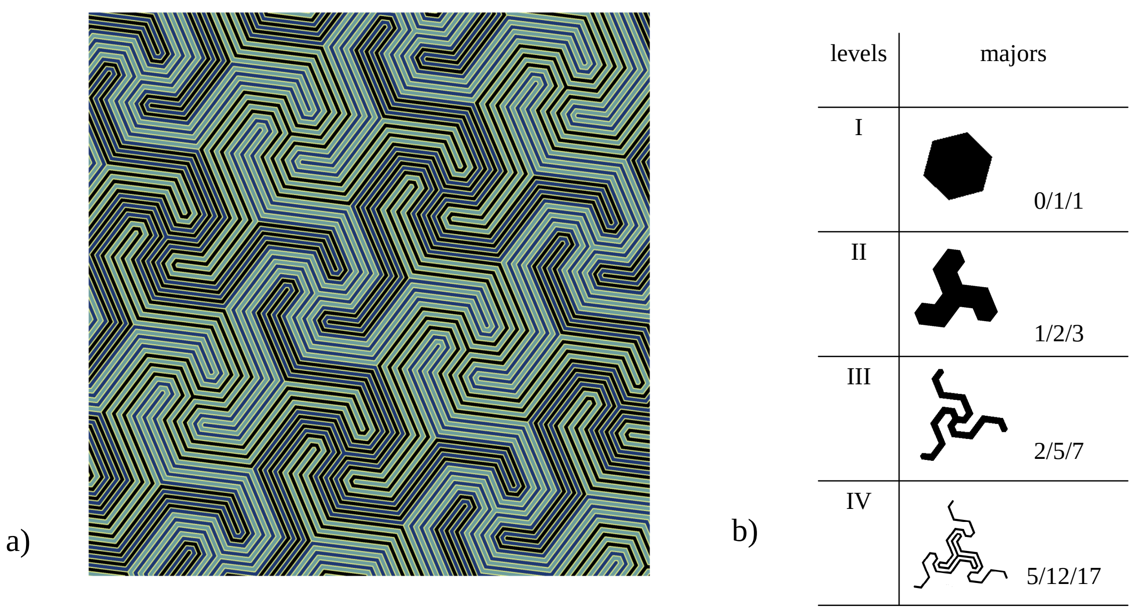


Figure 21 : *Stratified color mixing in P3*

its center as the final stripe path[9]. When spiral winding takes place around an axis lying in the center of a stripe path, only two out of the three colors are involved in this type of plane symmetry. At the highest level (III) this means that there are three mixtures: green mixes with black, green mixes with blue and black mixes with blue. In the lowest level (I) this means that these mixtures themselves are involved in mixing according to a mixing scheme of two out of three: green/black with green/blue, green/blue with blue/black and green/black with blue/black.

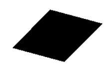
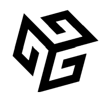
level	major
I	 0/1/1
II	 1/4/5

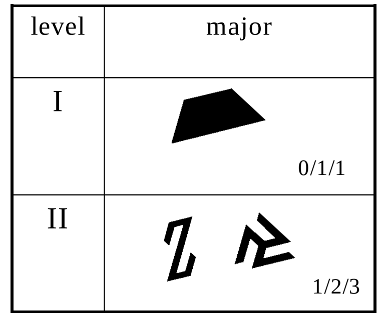
Figure 22: *More complex layered color mixture in P6.*

More complex is the color mixture in Figure 22. Here color mixture occurs because there is spiral winding around 2-fold axes of color bundles which are delivered by spiral winding around threefold axes in which different combinations of 3 out of 4 colors are involved.

More length values. The number of deflection lines in p6 can be enlarged from 2 to 4. For that system, let's take the composite ratio 6/13/19. Figure 23a shows the respective stripe pattern. There are two levels in spiral winding. The majors for the these levels are shown in Figure 23b. At level II there are two majors. The color mixture is wonderful. Six colors are occurring. At level II they are mixed according to a mixing scheme of 2 out of 6 around the perimeter of the Z-like major. And, also at this level, according to a mixing scheme of 3 out of 6 around the in the respective stripe path. perimeter of the three-legged swastika. At Level I the 2 out of 6 and the 3 out of 6 mixtures come together in a spiral formation around the perimeter of the trapezium. The series of *a*-values for the final stripe path is: $a_0 = 1$, $a_1 = 3$, $a_2 = 6$. When we enlarge the second *a*-value, the spiral winding around the perimeter of the trapezium goes a goes further (Figure 23c).

[9] The length value of the respective deflection line is even.

a)



b)

c)

Figure 23*: Layered color mixture in P6 , based on 4 length values in the system of deflection lines.*