

# Building Polyhedra from Polygons with Colored Edges

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## Abstract

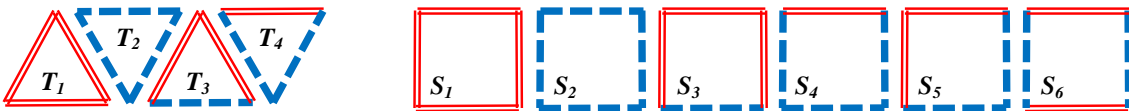
If we color the edges of a polyhedron and then break it apart into its polygonal faces we obtain a set of polygons with colored edges. We explore here the opposite problem: find sets of polygons with colored edges that can be assembled into various polyhedra by joining them along edges of the same color. We show how solutions to this problem can be used to design construction systems for building strong polyhedra models.

## Introduction

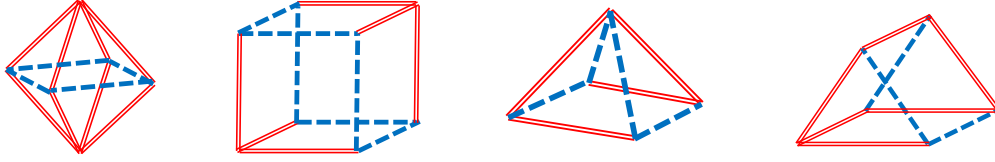
There are many geometric construction systems and toys that can be used to assemble polyhedra models by joining together polygonal faces on a common edge. One of the simplest systems consists of (roughly) polygonal shapes that can be connected by sliding them together along slits, cuts, or notches ([5], [7]). Most connection mechanisms allow polygons to be connected along any two edges of equal length, but here we are interested in systems that impose some restrictions on which edges can be joined together. In particular, we will consider an abstract model consisting of regular polygons with equal-length edges, each colored with one of two colors; the connections can be made only along edges that have the same color.

By definition, two such polygons are *equivalent* if we can map one to the other by a polygon transformation, i.e., a rotation, or, if the construction system being modeled allows “flipping” the polygonal pieces, a reflection. The equivalence class of a polygon will be called its *coloring*. A set  $C$  of colorings is *complete* for some class of polyhedra if all polyhedra in the class can be built using only polygons that have colorings in  $C$ . The set of all colorings of regular polygons with at most  $n$  sides is clearly complete for all polyhedra with such faces. The number of colorings grows exponentially with  $n$ ; the goal of this paper is to show that there exist smaller complete sets of colorings for various classes of polyhedra. (The set of trivial colorings with one color is also complete for all polyhedra, but we are interested here in colorings that use both colors.)

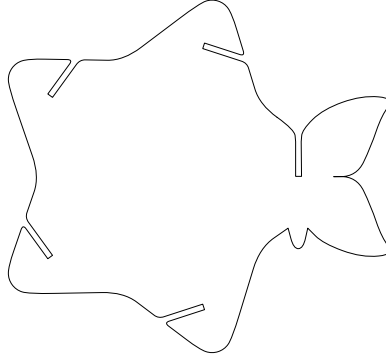
Referring to Figures 1 and 2, a tetrahedron, octahedron and icosahedron can be built using 4 equivalent triangles of any of the triangle colorings; the minimal complete coloring sets for these 3 polyhedra are thus  $\{T_1\}$ ,  $\{T_2\}$ ,  $\{T_3\}$ ,  $\{T_4\}$ . A cube can also be built with all faces having the same coloring as long as that coloring is not  $S_6$ ; minimal complete sets are  $\{S_1\}$ ,  $\{S_2\}$ ,  $\{S_3\}$ ,  $\{S_4\}$ ,  $\{S_5\}$ .  $\{T_3, S_1\}$ ,  $\{T_3, S_2\}$ ,  $\{T_3, S_5\}$  are minimal complete sets for the square pyramid, but  $\{T_3, S_3, S_4, S_6\}$  is not complete.  $\{T_3, T_4, S_3\}$ ,  $\{T_1, S_6\}$ ,  $\{T_3, S_3, S_5\}$ ,  $\{T_3, S_3, S_6\}$ ,  $\{T_3, S_5, S_6\}$ , etc., are minimal complete sets for the triangular prism, while  $\{T_3, T_4, S_6\}$ ,  $\{T_1, T_2, T_3, T_4, S_5\}$ ,  $\{T_3, S_3, S_4\}$ ,  $\{T_3, S_4, S_5\}$ , etc., are not complete. By counting the number of edges of the two colors we can see that  $\{T_1, T_3, S_2, S_4, S_5, S_6\}$  and  $\{T_1, T_3, T_4, S_2, S_4\}$  are not complete for the cuboctahedron.



**Figure 1 :** The triangle and square colorings (they are the same with or without reflections)



**Figure 2 :** Octahedron ( $T_3$ ), cube ( $S_5$ ), square pyramid ( $T_3, T_4, S_6$ ), triangular prism ( $T_3, S_3, S_5$ )



**Figure 3 :** Pentagonal piece with one reversed notch and a dodecahedron lantern built with such pieces

We will prove that  $\{T_3, T_4, S_5, S_6\}$  is complete for all polyhedra with triangular and square faces. We also conjecture that  $\{T_3, T_4, S_4, S_5\}$  is complete for convex polyhedra with triangular and square faces.

### Motivation

The motivation for this work comes from designing the **ITSPHUN** system of geometric shapes ([1]). The original design, like the systems mentioned above, has polygonal pieces with notches along their edges, all pointing in the same direction, either clockwise or counter-clockwise. This makes it easy to assemble a polyhedral model since the pieces, even when made from a relatively rigid material, can be gradually rotated into their final position. The uniform notch direction also makes it easy to disassemble a model by reversing the rotation, but has the disadvantage that pieces can sometimes rotate out of position by themselves causing the model to come apart. To counter this, we started experimenting with reversing the directions of some of the notches; a piece can thus have notches pointing in both directions, as illustrated in Figure 3. This makes the assembly process more difficult (but still possible due to the elasticity of the material) but prevents a piece from accidentally rotating out of position. Unless we have pieces with all possible combinations of notch directions, assembling a model now involves solving a puzzle since not every piece will fit at every location. We can represent the two notch directions by two “colors” and the need to minimize the number of different kinds of pieces while still allowing the construction of a large class of polyhedra models led us to the problem explored here.

### Graph Results

The original problem can be restated as an edge coloring problem on graphs by representing a polyhedron  $P$  by the graph  $G(P)$  which has as nodes the faces of  $P$  and as edges the edges of  $P$ . (If  $P$  is convex,  $G(P)$  is the *polyhedral graph* of the dual of  $P$ .)

Let  $R$  and  $B$  be two colors and let  $G = (V, E)$  be a graph with set of vertices  $V$  and set of edges

*E*. An (edge) *coloring* of  $G$  (not to be confused with the polygon coloring defined previously) is a map  $c : E \rightarrow \{R, B\}$ . For coloring  $c$  and  $v \in V$ , let  $d_X^c(v)$  be the number of  $X$  edges incident to  $v$ ,  $X \in \{R, B\}$  and  $s^c(v) = \langle d_R^c(v), d_B^c(v) \rangle$  the *color signature* of  $v$  in  $c$ . Note that color signatures do not distinguish between different ways in which polygon edge colors are cyclically ordered around a polyhedral face; we will discuss this in the next section.

Restating the problem described in the introduction in terms of graphs, a set of color signatures is *complete* for a class of graphs if any graph in the class has a coloring  $c$  such that  $s^c(v) \in S$  for all vertices  $v$ . We are looking for small sets of color signatures that are complete for certain graph classes.

Hilton and de Werra ([2], [6]) proved that all graphs have a *nearly equitable* coloring, i.e., a coloring  $c$  that satisfies  $|d_R^c(v) - d_B^c(v)| \leq 2$ . The following theorem strengthens this result with the additional restriction (2).

**Theorem 1.** Any graph  $G = (V, E)$  has a coloring  $c$  satisfying,  $\forall v \in V$ ,

$$|d_R^c(v) - d_B^c(v)| \leq 2 \quad (1)$$

$$d_R^c(v)d_B^c(v) \text{ even} \quad (2)$$

*Proof.* If  $G$  has an Eulerian circuit  $C$ , define  $c$  by traversing  $C$  and coloring the edges with alternate colors, except when leaving some vertex  $v$  on the last uncolored edge  $e$  incident to  $v$ ; in this case choose  $c(e)$  to satisfy (2). It is easy to see that  $c$  satisfies (1) at all nodes, and (2) at all nodes other than  $v_0$ , the starting and ending point of the traversal of  $C$ . But if  $d_R^c(v_0)$  is odd (and thus  $d_B^c(v_0)$  is also odd),  $v_0$  would be the only odd-degree vertex of the subgraph of  $G$  that contains only the  $R$  edges, contradiction, so (2) is also satisfied at  $v_0$ .

If  $G$  is connected but not Eulerian, then  $E$  can be covered by edge-disjoint, open paths whose end nodes have odd degree. Define  $c$  by traversing each such path and coloring the edges with alternate colors, except when leaving some *even-degree* vertex  $v$  on the last uncolored edge  $e$  incident to  $v$ ; in this case choose  $c(e)$  to satisfy (2).

If  $G$  is not connected, define  $c$  as above on each connected component. □

Having obtained a complete set of color signatures where the two colors are roughly balanced ( $\{\langle r, b \rangle \mid |r - b| \leq 2, rb \text{ even}\}$ ), we will present now a complete set for *convex* polyhedra where the number of edges of one color (say  $R$ ) incident to a vertex is as small as possible without becoming 0:  $\{\langle r, b \rangle \mid 1 \leq r \leq 2, b > 0\}$ .

Let  $d(v)$  be the degree of vertex  $v$ ,  $\mathcal{G}_{p,q}$  the set of graphs satisfying  $\forall v \in V, p \leq d(v) \leq q$  and  $\mathcal{S}_{p,q}$  the set of graphs  $G(V, E)$  such that  $G'(V, E') \in \mathcal{G}_{p,q}$  for some  $E' \subseteq E$ . Coloring all edges in  $E'$  with  $R$  and the rest with  $B$ , we obtain:

**Lemma 1.** Any graph in  $\mathcal{S}_{p,q}$  has a coloring  $c$  satisfying  $p \leq d_R^c(v) \leq q$  for all nodes  $v$ .

We are interested in  $\mathcal{S}_{p,q}$  with  $p > 0$  and  $q$  as small as possible. A graph that has a Hamiltonian path  $H$  is in  $\mathcal{S}_{1,2}$  (take  $E'$  to be the set of edges in  $H$ ). If it has a Hamiltonian circuit, then it is in  $\mathcal{S}_{2,2}$ . More general, if the graph has a vertex-disjoint path cover with nonempty paths then it is in  $\mathcal{S}_{1,2}$ ; if every path is a cycle, then it is in  $\mathcal{S}_{2,2}$  and if all paths have length 1 then it is in  $\mathcal{S}_{1,1}$ . A bipartite graph where one of the partitions has fewer than  $\frac{p}{p+q}$  of the nodes is not in  $\mathcal{G}_{p,q}$  and, since its subgraphs have the same property, not in  $\mathcal{S}_{p,q}$ .

**Lemma 2.** If a graph  $G$  has a closed walk visiting every vertex once or twice (a 2-walk) then  $G \in \mathcal{S}_{1,2}$ .

*Proof.* Let  $W$  be the closed 2-walk and let  $v$  be a vertex that has at least 3 incident edges in  $W$  (if no such vertex exists then  $W$  is a Hamiltonian cycle, thus  $G \in \mathcal{S}_{2,2}$ ). Let  $e$  be an edge incident to  $v$  traversed twice

by  $W$ , if such an edge exists, otherwise any edge in  $W$  incident to  $v$ . Color the edges of  $W$ , starting with  $e$ , alternately with  $B$  and  $R$  (if needed, change the color of an edge that is traversed twice; the color assigned by the second traversal is the final one). Color with  $B$  all edges not in  $W$ ; all vertices have now either one or two incident  $R$  edges.  $\square$

**Theorem 2.** *For any convex polyhedron  $P$ , the graph  $G(P)$  has a coloring  $c$  satisfying  $1 \leq d_R^c(v) \leq 2$  and  $d_B^c(v) > 0$  for all nodes  $v$ .*

*Proof.*  $G(P)$  is the polyhedral graph of the dual of  $P$ , thus by Steinitz's theorem is 3-connected and planar; therefore, according to Gao and Richter's theorem ([3], [4]), has a closed 2-walk. The result then follows from Lemmas 2 and 1.  $\square$

## Back to Polyhedra

Representing polygons as graph nodes abstracts away the sequence of edges around the polygon; for example, the two square colorings with two  $R$  edges and two  $B$  edges ( $S_5, S_6$  in Figure 1) are represented by the single (graph node) color signature  $\langle 2, 2 \rangle$ . Translating the graph results from the previous section back to polyhedra requires that we consider all possible arrangements of colored edges when defining a complete set of polygon colorings.

With this in mind, according to Theorem 1, we can build any polyhedron with 2 triangle colorings ( $T_3, T_4$ ), 2 squares ( $S_5, S_6$ ), 4 pentagons ( $P_5, P_6, P_7, P_8$  in Figure 4), 6 hexagons, etc. (The number of colorings for other polygons depends on whether reflections, or flipping of polygons, are allowed or not). For building convex polyhedra, a different complete set of colorings is obtained from Theorem 2: 2 triangles ( $T_3, T_4$ ), 3 squares ( $S_4, S_5, S_6$ ), 3 pentagons ( $P_4, P_6, P_8$ ), 4 hexagons, etc., and, in general,  $1 + \lfloor \frac{n}{2} \rfloor$  colorings of an  $n$ -sided polygon. (This number is the same whether we allow reflections or not.)

If the two colors represent notch directions, as described in the "Motivation" section, then flipping a polygon amounts to swapping the two colors. If this is possible, we can reduce the number of colorings given by Theorem 1 to 1 triangle, 2 squares, 2 pentagons, 3 hexagons, etc. For the complete set given by Theorem 2, the only reduction is from 2 to 1 in the number of triangle colorings.

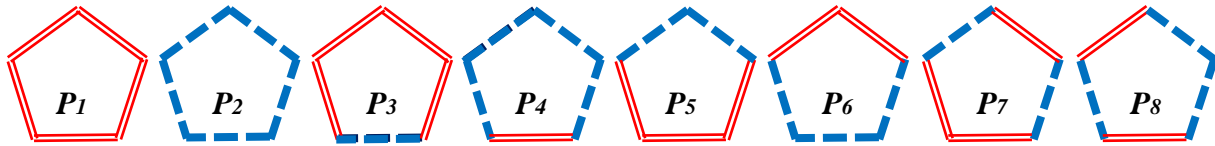
The following conjecture states that the complete set of color signatures given by Theorem 2 can be translated to an equal number of polygon colorings, avoiding the need to include all possible arrangements of colored edges.

**Conjecture 3.** *Any convex polyhedron with regular faces can be built with the set that has, for each polygon, a coloring with one  $R$  edge and a coloring with two adjacent  $R$  edges.*

If true, Conjecture 3 reduces the number of colorings in a complete set to just 2 for each polygon:  $T_3, T_4$  for triangles,  $S_4, S_5$  for squares,  $P_4, P_6$  for pentagons, etc.

## Conclusions and Future Work

We used graph-theoretical results to solve a practical problem in the design of systems for building polyhedral models from polygonal pieces. If each polygon edge is colored with one of two colors and the connections can be made only along edges of the same color, we showed that it is possible to avoid the combinatorial explosion of required polygon colorings. For pieces that connect through notches along their edges, colors can represent notch directions and our results can be used to design systems for building structurally strong polyhedra models using relatively few piece types.



**Figure 4 :** *The pentagon colorings (they are the same with or without reflections)*

Possible generalizations include construction systems where edges can be colored with more than two colors and systems with non-symmetrical (e.g., gendered) connections.

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