The Area Set Value Relationships in Atonal Music

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Abstract

The Area Set Value analysis is based on works of Rappaport (2005 and 2007), allowing one to understand the mathematical connections that exist between set-classes in atonal music. While traditional music theory distinguishes between 216 prime forms in sets with cardinality of 3 or larger, the ASV analysis allows for 48 values. This technique groups unrelated prime forms according to mathematical properties. The application of ASV is employed in Arnold Schoenberg's Suite for Piano, Op. 25, and String Quartet No. 4, Op. 37.

The ASV Groups

Using mathematics in music analysis allows one to examine a variety of repertoire. For instance, set theory uses graph theory, categorizes musical pitch-class sets, and defines their relationships in non-tonal context, while Neo-Riemannian analysis applies the notion of group theory to model intervals and chords [8]. This paper, based on the research of Rappaport (2005 and 2007), combines set and Neo-Riemannian analyses and examines the Area Set Values (ASVs), which are denoted with geometric representations on the unit circle in the Cartesian plane ($\mathbb{R}^2$) with a center at the origin and with radius of 1 [6]. The ASVs define chords of any cardinality as convex polygons. In his works, Rappaport uses integer partitions to generate scales [5]. As in Rappaport's research, the analysis in this paper will map pitch-classes onto polygons constructed from points equally spaced on the unit circle and calculate the area of the constructed polygons. Unlike Rappaport's research, this study groups all sets into unique categories and applies them to the musical analysis of Schoenberg's 12-tone technique.

The ASV employs integer notation, where 0 represents pitch-class C, 1 represents pitch-class C♯ or D♭, and so on. Likewise, the ASV incorporates enharmonic (F♯=G♭) and octave equivalences, which allow for cyclic representation of the pitch-classes in a closed musical cycle [4]. Therefore, the notion of modular arithmetic is applied, where 12 maps onto 0, 13 maps onto 1, and so on. There are likewise three nomenclatures (prime form, forte-number, and interval-class vector) under which the ASV analysis operates. Prime form is the most compact way to define a set that always begins with integer 0. For example, set {456} has a prime form of (012). Each prime form is defined by a forte-number (*x*-*y*), where *x* represents the cardinality of the set and *y* represents the index of a certain prime form among all prime forms of a given cardinality. For instance, the forte-number of all major and minor chords in musical pitch-space (037) is 3-11. Finally, each forte-number contains a unique interval-class vector (*icv*) – the intervallic make up of a pitch-class set, such as $<x_1\ x_2\ x_3\ x_4\ x_5\ x_6>$, where x_1 is the number of minor seconds and major sevenths, x_2 is the number of major seconds and minor sevenths, x_3 is the number of minor thirds and major sixths, x_4 is the number of major thirds and minor sixths, x_5 is the number of perfect fourths and perfect fifths, and x_6 is the number of tritones [7]. The ASV analysis functions under equal temperament tuning, which comprises of equal tuning of all twelve pitches of a modern chromatic scale, resulting in even distribution between any two neighboring notes [10]. The calculations for pitch-class locations, such as of (x_k, y_k), where each pitch-class is represented by a point on a unit circle, is as follows: $x_k = sin(\theta_k)$ and $y_k = cos(\theta_k)$, where $\theta_k = 2k\pi/12$, for k ranging from 0 to 11. Next, the points are connected, generating a polygon with n sides, where $3 \leq n \leq 12$. Trichords will produce triangles, tetrachords will produce quadrilaterals, and so on. The final step is to calculate the area of the shape. Figure 1 shows the unit circle, as used for the ASV analysis, and presents the smallest (3-1) and largest (12-1) ASVs. No side of a polygon can be larger than 2.

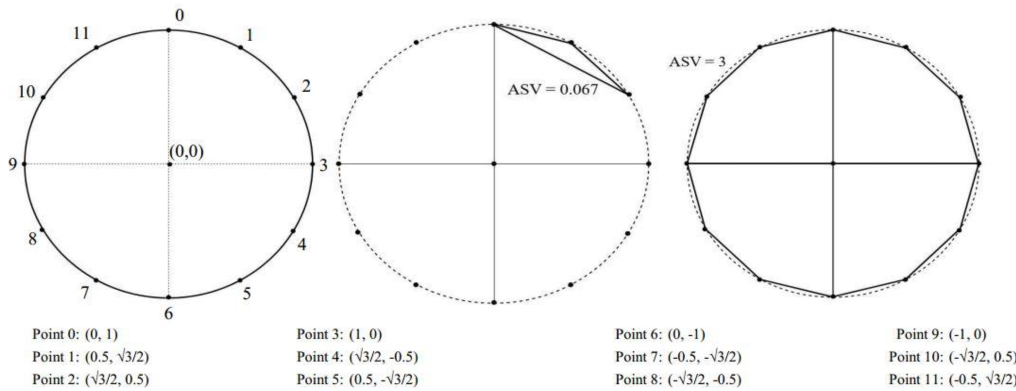

Figure 1: *The twelve pitch-class locations, ASV of 3-1, and ASV of 12-1.*

Table 1 provides the ASV analysis for chords with cardinality between 3 and 12, incorporating a total of 216 forte-numbers. These include 12 trichords, 29 tetrachords, 38 pentachords, 50 hexachords, 38 heptachords, 29 octachords, 12 nonachords, 6 decachords, 1 undecachord, and 1 dodecachord.

Table 1: *The ASV data for trichords through dodecachord.*

Type	Total Chords Analyzed	Total Areas	Unique Areas	Set // Smallest ASV	Set // Largest ASV	Total Average ASV
Trichord (3)	12	12	11	3-1 // 0.067	3-12 // 1.299	0.658
Tetrachord (4)	29	29	13	4-1 // 0.25	4-28 // 2	1.240
Pentachord (5)	38	38	12	5-1 // 0.567	5-35 // 2.299	1.719
Hexachord (6)	50	50	10	6-1 // 1	6-35 // 2.598	2.094
Heptachord (7)	38	38	7	7-1 // 1.5	7-35 // 2.665	2.366
Octachord (8)	29	29	5	8-1 // 2	8-28 // 2.732	2.576
Nonachord (9)	12	12	3	9-1 // 2.433	9-12 // 2.799	2.730
Decachord (10)	6	6	2	10-1 // 2.75	10-6 // 2.866	2.847
Undecachord (11)	1	1	1	11-1 // 2.933	11-1 // 2.933	2.933
Dodecachord (12)	1	1	1	12-1 // 3	12-1 // 3	3.000

The general trend of decreasing number of unique areas can be observed, as the cardinality of sets increases (with the exceptions of tetrachords). Also, as the cardinality increases, the smallest, largest, and average areas of each cardinality likewise increase, which leads to the notion of maximally even distributions that allow for the equal division of an octave, based on the interval content [3]. Examples of maximally even sets are tritones (2 tritones = 1 octave), augmented triads (3 major thirds = 1 octave), diminished seventh chords (4 minor thirds = 1 octave), and whole tone collections (6 major seconds = 1 octave). Furthermore, within each cardinality, the chromatic sets (3-1, 4-1, etc.) have the smallest ASVs, while the maximally even distributions (3-12, 4-28, etc.) have the largest ASVs. For each cardinality, the largest ASV comes from the polygon which is "closest" in total area to a regular polygon of the given order. Also, the "further" from a regular polygon the generated polygon is (based on its area), the smaller the ASV is, so that the smallest ASV is associated with the "least regular" of the polygons, which is the reason for $y = 1$ in forte numbers of smallest ASV sets in each cardinality group. Table 2 presents detailed analysis of all trichords, including its prime forms, *icv*, interval arrays (the directed interval distance between elements of the corresponding prime form), ASVs, as well as side lengths and angle values [2]. Three distinct observations can be made. First, the ASV increases as the forte number increases, except for 3-6, which is the subset of a whole tone collection. Second, the angles correspond to the values in the interval array and move in the increments of 15°, where an interval array of 1 represents 15°, an interval array of 2 represents 30°, and so on. Third, while different in its prime form and *icv*, 3-4 and 3-6 share identical ASVs. Furthermore, the uniqueness property of ASV analysis allows one to group two or more forte-numbers based on the mathematical area of their respective shapes. This demonstrates that the cardinality and the interval content does not reflect the area of the polygon that is being constructed. Therefore, while the total number of areas is tantamount to total chords analyzed, the number of unique areas is not and certain forte-numbers will generate identical areas. 37 of 48 ASVs contain more than 1 forte-number. The most common ASV is 2.366, occurring in 6-27, 6-Z28, 6-Z29, 6-30, 6-31, 6-Z45, 6-Z46, 6-Z47, 6-Z48, 6-Z49, 6-Z50, 8-2, 8-4, 8-5, and 8-6.

Table 2: *The ASV analysis of trichords*

Forte #	Prime Form	*icv*	Interval Array	ASV	Angles	Side 1	Side 2	Side 3
3-1	(012)	<210000>	1 - 1 - 10	0.067	15 - 15 - 150	0.518	0.518	1
3-2	(013)	<111000>	1 - 2 - 9	0.183	15 - 30 - 135	0.518	1	1.414
3-3	(014)	<101100>	1 - 3 - 8	0.317	15 - 45 - 120	0.518	1.414	1.732
3-4	(015)	<100110>	1 - 4 - 7	0.433	15 - 60 - 105	0.518	1.732	1.932
3-5	(016)	<100011>	1 - 5 - 6	0.500	15 - 75 - 90	0.518	1.932	2
3-6	(024)	<020100>	2 - 2 - 8	0.433	30 - 30 - 120	1	1	1.732
3-7	(025)	<011010>	2 - 3 - 7	0.683	30 - 45 - 105	1	1.414	1.932
3-8	(026)	<010101>	2 - 4 - 6	0.866	30 - 60 - 90	1	1.732	2
3-9	(027)	<010020>	2 - 5 - 5	0.933	30 - 75 - 75	1	1.932	1.932
3-10	(036)	<002001>	3 - 3 - 6	1.000	45 - 45 - 90	1.414	1.414	2
3-11	(037)	<001110>	3 - 4 - 5	1.183	45 - 60 - 75	1.414	1.732	1.932
3-12	(048)	<000300>	4 - 4 - 4	1.299	60 - 60 - 60	1.732	1.732	1.732

Application to Schoenberg

It is possible to interconnect the mathematical and musical theories behind the ways composers use sets in serial music. One of such connections can be seen in Schoenberg's use of 12-tone approach – a compositional technique that Schoenberg establishes to ensure the presentation of all 12 pitch-classes before any of them can undergo repetition [1]. An example, shown in Figure 2, can be seen in the second movement of Schoenberg's Suite for Piano, Op. 25, where the composer introduces the initial tone row (*row x*) in the third measure: {4-5-7-1-11-0-9-10-6-3-8-2}.

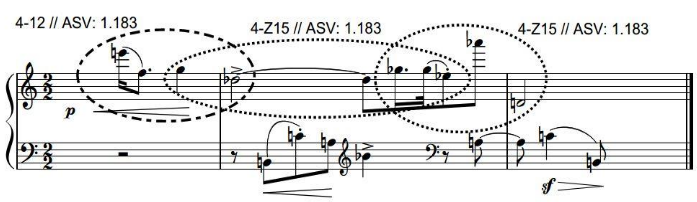

Figure 2: *Schoenberg, Suite for Piano, Op. 25, movement II, mm. 1-3.*

Two important subsets are derived from *row x*. The first four pitches of *row x* (x_1) are {4,5,7,1} that belong to the set 4-12 with prime form of (0236). The third and the fourth elements of x_1 (G and D♭) generate an interval of a tritone – the most unstable interval in Western music. The next set extracted from *row x* is 4-Z15 with prime form of (0146). This set initializes from the third and the fourth elements of x_1 and additionally includes the successive pitch-classes 6 and 3 in the upper staff, played by the right hand on the piano. Therefore, x_2 = {7,1,6,3}. Finally, a new subset can be extracted by amalgamating the third and the fourth elements of x_2 with successive pitch-classes 8 and 2 in the upper staff (the final two notes A♭ and D played by the right hand). Therefore, $x_3 = \{6,3,8,2\}$. The generation of one 4-12 and two 4-Z15 sets out of *row x* is not a coincidence and while x_2 begins with the tritone, x_1 and x_3 end on one. Both 4-12 and 4-Z15 have identical ASV of 1.183. The first movement of Arnold Schoenberg's String Quartet No. 4 presents another example of ASV connection to the twelve-tone technique. Schoenberg uses the tone row {2-1-9-10-5-3-4-0-8-7-6-11} in the opening six measures of Op. 37 (*row y*), shown in Figure 3 [9]. Sets in this work have discrete properties and appear in the tone row on more than one occasion. 3-4 (015) occurs in *row y* twice as {2,1,9} and {7,6,11}. 4-4 (0125) occurs in *row y* twice as {5,3,4,0} and {8,7,6,11}. The third tetrachord of *row y* is 4-7 (0145). While the notion of discreet tetrachords is evident, the ASV analysis demonstrates that the tetrachordal segmentation of *row y* generates three separate tetrachords with identical ASV value, as sets 4-4 and 4-7 have the ASV of 0.75.

Figure 3: *Schoenberg, String Quartet No. 4, Op. 37, movement I, mm. 1-6.*

The ASV can be used by music theorists, mathematicians, and performers. For theorists and mathematicians, the ASV allows for an alternate arrangement of pitch-class sets into 48 unique categories and generates significant patterns that can be related to atonal music. Outside of set and transformational theories, the ASV analysis incorporates trigonometric properties that allow one to understand how the intervallic content of each set reflects the side lengths, the angles, and the total areas of the polygons. More importantly, the ASV explains the similarities that exist in mathematical properties of two or more sets that are unrelated, based on their intervallic content. For performers, the ASV can be used to generate and shape their own musical interpretation. An identical ASV in two or more pitch-class sets is a unique property and can be emphasized musically with dynamics, tempo, and articulation.

References

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